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MASSLESS SPIN 3/2 FIELD, SPHERICAL SOLUTIONS, ELIMINATING OF THE GAUGE DEGREES OF FREEDOM

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Abstract. Relativistic system for a vector-bispinior describing a massless spin 3/2 field is studied in the spherical coordinates of Minkowski space. Presentation of the equation with the use of the covariant Levi-Civita tensor exhibits existence of the gauge solutions in the form of the covariant 4-gradient of an arbitrary bispinor. Substitution for 16-component field function is based on the use of Wigner functions, it assumes diagonalization of the operators of energy, square and third projection of the total angular momentum, and space reflection. We derive radial system for eight independent functions. General structure of the spherical gauge solutions is specified, and it is demonstrated that the gauge radial functions satisfy the derived system. It is proved that the general system reduces to two couples of independent 2-nd order and nonhomogeneous differential equations, their particular solutions may be found with the use of the gauge solutions. The corresponding homogeneous equations have one the same form, they have three regular singularities and one irregular of the rank 2. Frobenius types solutions for this equation have been constructed, and the structure of the involved power series with 4-term recurrent relations sre studied. Six remaining radial functions may be straightforwardly found by means of the simple algebraic relations. Thus, we have constructed two types of solutions with opposite parities which do not contain gauge constituents.

Keywords: spin 3/2, massless field, gauge symmetry, tetrad formalism, Minkowski space, spherical coordinates, Wigner *D*-functions, exact solutions, exclusion of the gauge degrees of freedom

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БЕЗМАССОВОЕ ПОЛЕ СО СПИНОМ 3/2: СФЕРИЧЕСКИЕ РЕШЕНИЯ И УСТРАНЕНИЕ КАЛИБРОВОЧНЫХ СТЕПЕНЕЙ СВОБОДЫ

(Представлено членом-корреспондентом Д. С. Могилевцевым)

Аннотация. Релятивистская система уравнений для вектор-биспинора, описывающего безмассовую частицу со спином 3/2, исследуется в сферической системе координат и соответствующей тетраде пространства Минковского. Представление волнового уравнения с использованием тензора Леви–Чивита выявляет существование калибровочных решений в виде 4-дивергенции от произвольного биспинора. Подстановка для 16-компонетной полевой функции основана на использовании функций Вигнера, она предполагает диагонализацию четырех операторов: энергии, квадрата и третьей проекции полного углового момента, а также оператора пространственного отражения. После разделения переменных выведена система из 8 радиальных уравнений. Детализируется общая структура калибровочных сферически симметричных решений, показывается, что эти радиальные функции обращают в тождества все 8 уравнений общей системы. Показывается, что общая система приводится к двум парам неоднородных дифференциальных уравнений второго порядка, их частные решения построены на основе использования калибровочных решений специального вида. Соответствующие однородные уравнения имеют одну и ту же структуру с тремя регулярными особыми точками и одной нерегулярной ранга 2. Построены их решения, исследована структура входящих в них степенных рядов с 4-членными рекуррентными соотношениями. Таким образом, построены два независимых класса решений с противоложными четностями, которые не содержат калибровочных компонент.

Ключевые слова: спин 3/2, безмассовое поле, калибровочная симметрия, тетрадный формализм, пространство Минковского, сферические координаты, функции Вигнера, точные решения, исключение калибровочных степеней свободы

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Introduction. The theory of spin 3/2 particle is attracted steady interest after the seminal investigation by Pauli–Fierz [1; 2] and Rarita–Schwinger [3], this subject has a long history (see in [4–20]). Let us recall the most significant aspects of this theory. First of all, it is the problem of choosing the initial equations. The most consistent is the approach based on the Lagrangian formalism and the first order system equations for multi-component field. However the most of investigations are performed with the use of the second order equations, so is the basic Pauli–Fierz approach. Applying the first order formalism ensures the correct solving of the problem of independent degrees of freedom in presence of external fields; for instance see in [18]). The great attention was given to existence in this theory solutions which correspond to states when the particle moves with velocity greater than the light velocity. Finally a separate interest has a massless case for spin 3/2 field, when as shown by Pauli and Fierz there exists specific gauge symmetry: the 4-gradient of arbitrary bispinor function provides us with solutions for the massless field equation for instance, see in [18]. Similar gauge symmetry arises also for massless spin 2 theory referring to graviton. These gauge solutions do not contribute in physically observable quantities such as the energy and linear momentum. In the present paper we examine the problem of spherical solutions for the 16-component system of equations describing a massless spin 3/2 particle in Minkowski space, we specify gauge solutions with spherical symmetry and construct solutions which do not contain the gauge components.

Massless spin 3/2 particle, general theory. We start with the generally covariant equation for a massless spin 3/2 particle (see notations in [20])

$$\frac{i}{2}\gamma^5 \varepsilon_\rho{}^{\mu\alpha\beta}(x)\gamma_\mu(x)[\nabla_\alpha + \Gamma_\alpha]\Psi_\beta = 0. \tag{1}$$

It is readily proved that for many space-time models, eq. (1) has the class of gradient type solutions

$$\Psi^G_\beta(x) = D_\beta \Psi(x), \quad D_\beta = (\nabla_\beta + \Gamma_\beta),$$

where Ψ is an arbitrary bispinor field. Indeed, substituting this gauge solution into eq. (1), we obtain

$$\frac{i}{2}\gamma^5 \varepsilon_\rho{}^{\mu\alpha\beta}\gamma_\mu(x) D_\alpha D_\beta \Psi(x) = \frac{i}{4}\gamma^5 \varepsilon_\rho{}^{\mu\alpha\beta}\gamma_\mu(x)[D_\alpha D_\beta - D_\beta D_\alpha]\Psi(x). \tag{2}$$

Due to identity (see in [18])

$$\frac{i}{4}\gamma^5 \varepsilon_\rho{}^{\mu\alpha\beta}\gamma_\mu(x)[D_\alpha D_\beta - D_\beta D_\alpha]\Psi(x) = [R_{\alpha\beta}(x) - \frac{1}{2}R(x)\, g_{\alpha\beta}(x)]\gamma^\beta \Psi(x),$$

we conclude that for any space-time, which is a solution of Einstein equations with vanishing energy-momentum tensor, relation (2) vanishes identically as well.

Below it will be convenient to use the field function with tetrad vector index $\Psi_l(x)$:

$$\Psi_\beta(x) = e^{(l)}_\beta(x)\Psi_l(x), \quad \Psi_l(x) = e^\beta_{(l)}(x)\Psi_\beta(x).$$

Correspondingly, eq. (1) is transformed to the form (Ricci rotation coefficients are used):

$$\frac{i}{2}\gamma^5 \varepsilon_k{}^{bcd}\gamma_b \left[e^\alpha_{(c)}\partial_\alpha + \frac{1}{2}(\sigma^{nm} \otimes I + I \otimes j^{nm})\,\gamma_{[nm]c} \right]_d^{\;l} \Psi_l = 0, \tag{3}$$

where $\Psi(x)$ stands for the matrix with two indices, bispinor's A and vector's l. We will use the short form of equation

$$\frac{i}{2}\gamma^5 \varepsilon_k{}^{bcd}\gamma_b (D_c)_d{}^l \Psi_l = 0 \;\; \Rightarrow \;\; \frac{i}{2}\gamma^5 \varepsilon_k{}^{bcd}\gamma_b (D_c \Psi)_d = 0, \tag{4}$$

where we use the notation $D_c = e^{\alpha}_{(c)}\partial_\alpha + \frac{1}{2}(\sigma^{nm} \otimes I + I \otimes j^{nm})\,\gamma_{[nm]c}$.

When constructing spherically symmetric solutions for massive spin 3/2 particle in [8], transition from vector index in Cartesian basis Ψ to cyclic one $\bar{\Psi}$ was used. In such a cyclic basis it was found the general substitution for field function related to diagonalization of the square and third projection of the total angular momentum:

$$\bar{\Psi} = e^{-i\varepsilon t} \begin{vmatrix} f_0(r)D_{-1/2} & f_1(r)D_{-3/2} & f_2(r)D_{-1/2} & f_3(r)D_{+1/2} \\ g_0(r)D_{+1/2} & g_1(r)D_{-1/2} & g_2(r)D_{+1/2} & g_3(r)D_{+3/2} \\ h_0(r)D_{-1/2} & h_1(r)D_{-3/2} & h_2(r)D_{-1/2} & h_3(r)D_{+1/2} \\ d_0(r)D_{+1/2} & d_1(r)D_{-1/2} & d_2(r)D_{+1/2} & d_3(r)D_{+3/2} \end{vmatrix}, \tag{5}$$

where symbols D designate the Wigner functions, $D_\sigma = D^j_{-m,\sigma}(\phi, \theta, 0)$; $j = 1/2, 3/2, 5/2, \ldots$. For more details see in [21]. It should be noted that at the minimal value $j = 1/2$ the above substitution is simplified accordingly to relations: $j = 1/2,\ f_1 = 0,\ g_3 = 0,\ h_1 = 0,\ d_3 = 0$. Connection between Cartesian Ψ and cyclic $\bar{\Psi}$ bases is determined by the formulas

$$\bar{\Psi} = (I \otimes U)\Psi, \quad U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad \Psi = (I \otimes U^{-1})\bar{\Psi}, \;\; U^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

Let us transform the wave equation from Cartesian basis to the cyclic one. With the use of notations

$$U_{ps}(D_c)_s{}^l U^{-1}_{ln} = (\bar{D}_c)_p{}^n, \quad U_{sk}\varepsilon_k{}^{bcd}U^{-1}_{dp} = U_{sk}\varepsilon_k{}^{bcd}U^{-1}_{dp} = \bar{\varepsilon}_s{}^{bcp},$$

the wave equation in cyclic basis is presented as follow

$$\frac{i}{2}\gamma^5 \gamma_b \bar{\varepsilon}_s{}^{bcp}(\bar{D}_c)_p{}^n \bar{\Psi}_n = 0. \tag{6}$$

Let us specify expression for components of the new Levi–Civita tensor $\bar{\varepsilon}_s{}^{bcp} = U_{sk}\varepsilon_k{}^{bcd}U^{-1}_{dp}$. It is convenient to apply the matrix notations, $\mu^{[bc]}_{kd} = \varepsilon_k{}^{bcd}$, $\bar{\mu}^{[bc]}_{sp} = \bar{\varepsilon}_s{}^{bcp}$, then we have the rule

$$\bar{\mu}^{[bc]}_{sp} = U_{sk}\,\mu^{[bc]}_{kd}\,U^{-1}_{dp};$$

the square brackets mark antisymmetry in two indices. Further we find explicit form for matrices $\bar{\mu}^{[bc]}_{sp}$:

$$\bar{\mu}^{[01]}_{sp} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ 0 & 0 & -i & 0 \end{vmatrix}, \quad \bar{\mu}^{[02]}_{sp} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \bar{\mu}^{[03]}_{sp} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{vmatrix},$$

$$\bar{\mu}^{[23]}_{sp} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \bar{\mu}^{[31]}_{sp} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -i & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad \bar{\mu}^{[12]}_{sp} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

Let us turn to eq. (6) written as

$$\frac{i}{2}\gamma^5\{(\gamma^1\otimes\bar{\mu}^{[01]}+\gamma^2\otimes\bar{\mu}^{[02]}+\gamma^3\otimes\bar{\mu}^{[03]})\bar{D}_0\bar{\Psi}+(\gamma^0\otimes\bar{\mu}^{[03]}+\gamma^1\otimes\bar{\mu}^{[31]}-\gamma^2\otimes\bar{\mu}^{[23]})\bar{D}_3\bar{\Psi}+$$
$$+(\gamma^0\otimes\bar{\mu}^{[01]}+\gamma^2\otimes\bar{\mu}^{[12]}-\gamma^3\otimes\bar{\mu}^{[31]})\bar{D}_1\bar{\Psi}+(\gamma^0\otimes\bar{\mu}^{[02]}+\gamma^3\otimes\bar{\mu}^{[23]}-\gamma^1\otimes\bar{\mu}^{[12]})D_2\bar{\Psi}\}=0. \quad (7)$$

Separation of the variables. Let us consider eq. (7) in the spherical coordinates

$$x^\alpha=(t,r,\theta,\phi)\quad dS^2=dt^2-dr^2-r^2d\theta^2-\sin^2\theta d\phi^2$$

and corresponding tetrad. For components of the operator $\bar{D}_c$ we obtain the following expressions

$$\bar{D}_0=\partial_t,\quad \bar{D}_3=\partial_r,\quad \bar{D}_1=\frac{1}{r}\partial_\theta+\frac{1}{r}(\sigma^{31}\otimes I+I\otimes\bar{j}^{31}),$$
$$\bar{D}_2=\frac{1}{r}(\sigma^{32}\otimes I+I\otimes\bar{j}^{32})+\frac{1}{r}\frac{\partial_\phi+\cos\theta(\sigma^{12}\otimes I+I\otimes\bar{j}^{12})}{\sin\theta}.$$

The general substitution for field function was given in (5). In [4; 5], restrictions for radial functions corresponding to diagonalization of the spatial reflection operator were found:

$$d_0=\delta f_0, d_1=\delta f_3,\ d_2=\delta f_2, d_3=\delta f_1,\ \ h_0=\delta g_0, h_1=\delta g_3, h_2=\delta g_2, h_3=\delta g_1,\quad \delta=+1,-1 \quad (8)$$

so we have only 8 independent functions. Below when separating the variables we will apply the known recurrent formulas for Wigner functions [21]:

$$\partial_\theta\, D_{+1/2}=\frac{1}{2}(a\,D_{-1/2}-b\,D_{+3/2}),\quad \partial_\theta\, D_{-1/2}=\frac{1}{2}(b\,D_{-3/2}-a\,D_{+1/2}),$$

$$\frac{1}{\sin\theta}\left(-m-\frac{1}{2}\cos\theta\right)D_{+1/2}=\frac{1}{2}(-a\,D_{-1/2}-b\,D_{+3/2}),\ \frac{1}{\sin\theta}\left(-m+\frac{1}{2}\cos\theta\right)D_{-1/2}=\frac{1}{2}(-b\,D_{-3/2}-a\,D_{+1/2}),$$

$$\frac{1}{\sin\theta}\left(-m-\frac{3}{2}\cos\theta\right)D_{+3/2}=\frac{1}{2}(-b\,D_{+1/2}-c\,D_{+5/2}),\ \frac{1}{\sin\theta}\left(-m+\frac{3}{2}\cos\theta\right)D_{-3/2}=\frac{1}{2}(-c\,D_{-5/2}-b\,D_{-1/2}),$$

where $a=j+1/2,\quad b=\sqrt{(j-1/2)(j+3/2)},\quad c=\sqrt{(j-3/2)(j+5/2)}$.

After rather long calculations from eq. (7) we derive the system of 8 radial equations

$$\sqrt{2}\frac{d}{dr}g_1+\frac{1}{r}\left(f_2+\frac{3}{\sqrt{2}}g_1\right)+\frac{1}{\sqrt{2}r}(bf_1-af_3+a\sqrt{2}g_2)=0,$$

$$\sqrt{2}\frac{d}{dr}f_3+\frac{1}{r}\left(g_2+\frac{3}{\sqrt{2}}f_3\right)+\frac{1}{\sqrt{2}r}(-ag_1+bg_3+a\sqrt{2}f_2)=0,$$

$$-i\varepsilon f_1+\frac{d}{dr}f_1+\frac{1}{r}f_1+\frac{1}{\sqrt{2}r}(bf_2+bf_0)=0,$$

$$-i\varepsilon(\sqrt{2}f_2-g_1)+\left(-\sqrt{2}\frac{d}{dr}f_0+\frac{d}{dr}g_1\right)-\frac{1}{r}\left(\frac{1}{\sqrt{2}}(f_0-f_2)-g_1\right)+\frac{1}{\sqrt{2}r}(ag_2-ag_0)=0,$$

$$-i\varepsilon\sqrt{2}g_1+\frac{1}{r}\left(f_0-\frac{1}{\sqrt{2}}g_1\right)+\frac{1}{\sqrt{2}r}(-bf_1+af_3+a\sqrt{2}g_0)=0,$$

$$-i\varepsilon\sqrt{2}f_3+\frac{1}{r}\left(g_0+\frac{1}{\sqrt{2}}f_3\right)+\frac{1}{\sqrt{2}r}(-ag_1+bg_3+a\sqrt{2}f_0)=0,$$

$$-i\varepsilon(\sqrt{2}g_2-f_3)+\left(-\sqrt{2}\frac{d}{dr}g_0-\frac{d}{dr}f_3\right)-\frac{1}{r}\left(\frac{1}{\sqrt{2}}(g_0+g_2)+f_3\right)+\frac{1}{\sqrt{2}r}(-af_2-af_0)=0,$$

$$-i\varepsilon g_3 - \frac{d}{dr} g_3 - \frac{1}{r} g_3 + \frac{1}{\sqrt{2}r}(-bg_2 + bg_0) = 0. \tag{9}$$

Let us introduce the new combinations of functions

$$F_0 = f_0 + g_0, \quad G_0 = f_0 - g_0, \quad F_1 = f_1 + g_3, \quad G_1 = f_1 - g_3,$$

$$F_2 = f_2 + g_2, \quad G_2 = f_2 - g_2, \quad F_3 = f_3 + g_1, \quad G_3 = f_3 - g_1. \tag{10}$$

Besides, to simplify equations let us separate the simple multiplier $1/r$ at all functions (for simplicity we preserve the same designations for new radial functions):

$$f \Rightarrow \frac{1}{r} f, \quad \left(\frac{d}{dr} + \frac{1}{r}\right)\frac{1}{r} f \Rightarrow \frac{1}{r}\frac{d}{dr} f.$$

In this way we derive the system

$$+\sqrt{2}\left(\frac{d}{dr} + \frac{1}{2r} - \frac{a}{2r}\right)F_3 + \left(\frac{1}{r} + \frac{a}{r}\right)F_2 + \frac{b}{\sqrt{2}r}F_1 = 0,$$

$$-\sqrt{2}\left(\frac{d}{dr} + \frac{1}{2r} + \frac{a}{2r}\right)G_3 + \left(\frac{1}{r} - \frac{a}{r}\right)G_2 + \frac{b}{\sqrt{2}r}G_1 = 0,$$

$$-i\varepsilon\sqrt{2}F_3 + \left(\frac{1}{r} + \frac{a}{r}\right)F_0 + \left(\frac{1}{\sqrt{2}r} + \frac{a}{\sqrt{2}r}\right)G_3 - \frac{b}{\sqrt{2}r}G_1 = 0,$$

$$i\varepsilon\sqrt{2}G_3 + \left(\frac{1}{r} - \frac{a}{r}\right)G_0 + \left(-\frac{1}{\sqrt{2}r} + \frac{a}{\sqrt{2}r}\right)F_3 - \frac{b}{\sqrt{2}r}F_1 = 0,$$

$$-i\varepsilon F_1 + \frac{d}{dr}G_1 + \frac{b}{\sqrt{2}r}G_2 + \frac{b}{\sqrt{2}r}F_0 = 0, \quad -i\varepsilon G_1 + \frac{d}{dr}F_1 + \frac{b}{\sqrt{2}r}F_2 + \frac{b}{\sqrt{2}r}G_0 = 0,$$

$$-i\varepsilon\sqrt{2}F_2 + i\varepsilon F_3 - \sqrt{2}\left(\frac{d}{dr} - \frac{1}{2r} + \frac{a}{2r}\right)F_0 - \frac{d}{dr}G_3 + \left(\frac{1}{\sqrt{2}r} - \frac{a}{\sqrt{2}r}\right)G_2 = 0,$$

$$-i\varepsilon\sqrt{2}G_2 - i\varepsilon G_3 - \sqrt{2}\left(\frac{d}{dr} - \frac{1}{2r} - \frac{a}{2r}\right)G_0 + \frac{d}{dr}F_3 + \left(\frac{1}{\sqrt{2}r} + \frac{a}{\sqrt{2}r}\right)F_2 = 0. \tag{11}$$

In the next section we will find explicit form for gauge solutions of the system of equations for massless spin 3/2 particle. The corresponding radial functions must satisfy identically the general system (11). Besides, the gauge solutions will be important further in solving the general radial system.

Gradient type solutions. Let Ψ be an arbitrary spherically symmetric bispinor field. We may assume that this bispinor is some solution of the Dirac equation, however this requirement is not necessary. As we will see later only general structure of such bispinors is substantial:

$$\Psi = e^{-i\varepsilon t}\begin{vmatrix} K_1 D_{-1/2} \\ K_2 D_{+1/2} \\ K_3 D_{-1/2} \\ K_4 D_{+1/2} \end{vmatrix}.$$

The gradient solutions are determined by the relation $\Psi_c^g = D_c\Psi$, $D_c = e^{\alpha}_{(c)}(\partial_\alpha + \Gamma_\alpha)$. So we get

$$\Psi_0^g = D_0\Psi = -i\varepsilon\begin{vmatrix} K_1 D_{-1/2} \\ K_2 D_{+1/2} \\ K_3 D_{-1/2} \\ K_4 D_{+1/2} \end{vmatrix}, \quad \Psi_3^g = D_3\Psi\begin{vmatrix} K'_1 D_{-1/2} \\ K'_2 D_{+1/2} \\ K'_3 D_{-1/2} \\ K'_4 D_{+1/2} \end{vmatrix}, \quad \Psi_1^g = D_1\Psi = \frac{1}{r}\begin{vmatrix} K_1\partial_\theta D_{-1/2} \\ K_2\partial_\theta D_{+1/2} \\ K_3\partial_\theta D_{-1/2} \\ K_4\partial_\theta D_{+1/2} \end{vmatrix} + \frac{1}{2r}\begin{vmatrix} -K_2 D_{+1/2} \\ K_1 D_{-1/2} \\ -K_4 D_{+1/2} \\ K_3 D_{-1/2} \end{vmatrix},$$

$$\Psi_2^g = D_2\Psi = \frac{i}{2r}\begin{vmatrix} K_2 D_{+1/2} \\ K_1 D_{-1/2} \\ K_4 D_{+1/2} \\ K_3 D_{-1/2} \end{vmatrix} - \frac{i}{r}\frac{1}{\sin\theta}\begin{vmatrix} K_1(-m+\cos\theta\frac{1}{2})D_{-1/2} \\ K_2(-m-\cos\theta\frac{1}{2})D_{+1/2} \\ K_3(-m+\cos\theta\frac{1}{2})D_{-1/2} \\ K_4(-m-\cos\theta\frac{1}{2})D_{+1/2} \end{vmatrix}.$$

Taking in mind the recurrent formulas for Wigner functions, we get expressions for $\Psi_{(1)}^g$ and $\Psi_{(2)}^g$:

$$\Psi_1^g = \frac{1}{r}\begin{vmatrix} K_1(\beta D_{-3/2} - \alpha D_{+1/2}) \\ K_2(\alpha D_{-1/2} - \beta D_{+3/2}) \\ K_3(\beta D_{-3/2} - \alpha D_{+1/2}) \\ K_4(\alpha D_{-1/2} - \beta D_{+3/2}) \end{vmatrix} + \frac{1}{2r}\begin{vmatrix} -K_2 D_{+1/2} \\ K_1 D_{-1/2} \\ -K_4 D_{+1/2} \\ K_3 D_{-1/2} \end{vmatrix}, \quad \Psi_2^g = -\frac{i}{r}\begin{vmatrix} K_1(-\beta D_{-3/2} - \alpha D_{+1/2}) \\ K_2(-\alpha D_{-1/2} - \beta D_{+3/2}) \\ K_3(-\beta D_{-3/2} - \alpha D_{+1/2}) \\ K_4(-\alpha D_{-1/2} - \beta D_{+3/2}) \end{vmatrix} + \frac{i}{2r}\begin{vmatrix} K_2 D_{+1/2} \\ K_1 D_{-1/2} \\ K_4 D_{+1/2} \\ K_3 D_{-1/2} \end{vmatrix},$$

where $2\alpha = (j+1/2) = a,\ 2\beta = \sqrt{(j-1/2)(j+3/2)} = b.$ Let us combine the components Ψ_a^g so that to get the quantities referring to cyclic basis:

$$\bar{\Psi}_0^g = \Psi_0^g = -i\varepsilon\begin{vmatrix} K_1 D_{-1/2} \\ K_2 D_{+1/2} \\ K_3 D_{-1/2} \\ K_4 D_{+1/2} \end{vmatrix}, \quad \bar{\Psi}_2^g = \Psi_3 = \begin{vmatrix} K'_1 D_{-1/2} \\ K'_2 D_{+1/2} \\ K'_3 D_{-1/2} \\ K'_4 D_{+1/2} \end{vmatrix},$$

$$\bar{\Psi}_1^g = \frac{1}{\sqrt{2}}(-\Psi_1^g + i\Psi_2^g) = \frac{1}{r}\frac{1}{\sqrt{2}}\begin{vmatrix} -K_1 2\beta D_{-3/2} \\ -K_2 2\alpha D_{-1/2} \\ -K_3 2\beta D_{-3/2} \\ -K_4 2\alpha D_{-1/2} \end{vmatrix} + \frac{1}{r}\frac{1}{\sqrt{2}}\begin{vmatrix} 0 \\ -K_1 D_{-1/2} \\ 0 \\ -K_3 D_{-1/2} \end{vmatrix}, \tag{12}$$

$$\bar{\Psi}_3^g = \frac{1}{\sqrt{2}}(\Psi_1^g + i\Psi_2^g) = \frac{1}{r}\frac{1}{\sqrt{2}}\begin{vmatrix} -K_1 2\alpha D_{+1/2} \\ -K_2 2\beta D_{+3/2} \\ -K_3 2\alpha D_{+1/2} \\ -K_4 2\beta D_{+3/2} \end{vmatrix} + \frac{1}{r}\frac{1}{\sqrt{2}}\begin{vmatrix} -K_2 D_{+1/2} \\ 0 \\ -K_4 D_{+1/2} \\ 0 \end{vmatrix}.$$

It is known that the space reflection operator for bispinor field permits us to divide solutions in two types, this is reached by imposing the following restrictions $\Delta = \pm 1,\ K_4 = \Delta K_1,\ K_3 = \Delta K_2$. Then instead of (12) we get

$$\bar{\Psi}_0^g = \begin{vmatrix} -i\varepsilon K_1 A \\ -i\varepsilon K_2 B \\ -i\varepsilon\Delta K_2 A \\ -i\varepsilon\Delta K_1 B \end{vmatrix}, \bar{\Psi}_1^g = \frac{1}{\sqrt{2}r}\begin{vmatrix} -2\beta K_1 C \\ -(2\alpha K_2 + K_1)A \\ -2\beta\Delta K_2 C \\ -\Delta(2\alpha K_1 + K_2)A \end{vmatrix}, \bar{\Psi}_2^g = \begin{vmatrix} K'_1 A \\ K'_2 B \\ \Delta K'_2 A \\ \Delta K'_1 B \end{vmatrix}, \bar{\Psi}_3 = \frac{1}{\sqrt{2}r}\begin{vmatrix} -(2\alpha K_1 + K_2)B) \\ -2\beta K_2 D \\ -\Delta(2\alpha K_2 + K_1)B \\ -2\beta\Delta K_1 D \end{vmatrix}.$$

The general structure of this gauge solution may be presented as follows

$$\bar{f}_0 = -i\varepsilon K_1,\ \bar{g}_0 = -i\varepsilon K_2,\ \bar{f}_1 = -\frac{2\beta}{\sqrt{2}r}K_1,\ \bar{g}_1 = -\frac{1}{\sqrt{2}r}(2\alpha K_2 + K_1),$$
$$\bar{f}_2 = K'_1,\ \bar{g}_2 = K'_2,\ \bar{f}_3 = -\frac{1}{\sqrt{2}r}(2\alpha K_1 + K_2),\ \bar{g}_3 = -\frac{1}{\sqrt{2}r}2\beta K_2. \tag{13}$$

By direct calculations we can prove that eqs. (9) are satisfied identically by functions from (13). It should be emphasized that in this proving the explicit form of the radial functions from (13) is not used. After translating the formulas (13) to the variables defined by (10) we obtain (remember that in all functions we have separated the multiplier $1/r$)

$$\bar{F}_0 = -i\varepsilon(K_1+K_2),\;\; \bar{F}_1 = -\frac{2\beta}{\sqrt{2}r}(K_1+K_2),\;\; \bar{F}_2 = \left(\frac{d}{dr}-\frac{1}{r}\right)(K_1+K_2),\;\; \bar{F}_3 = -\frac{2\alpha+1}{\sqrt{2}r}(K_1+K_2),$$

$$G_0 = -i\varepsilon(K_1-K_2),\;\; \bar{G}_1 = -\frac{2\beta}{\sqrt{2}r}(K_1-K_2),\;\; \bar{G}_2 = \left(\frac{d}{dr}-\frac{1}{r}\right)(K_1-K_2),\;\; \bar{G}_3 = -\frac{2\alpha-1}{\sqrt{2}r}(K_1-K_2). \quad (14)$$

Also, we can verify that eqs. (11) are satisfied identically by eight functions from (14).

Solving the system of radial equations. Let us turn to the system (11). It is convenient to simplify explicit form of these equations by changing the variables (for simplicity the notations of them we preserve the same):

$$\frac{F_0}{\sqrt{2}} \Rightarrow F_0,\; \frac{G_0}{\sqrt{2}} \Rightarrow G_0,\; \frac{F_2}{\sqrt{2}} \Rightarrow F_2,\; \frac{G_2}{\sqrt{2}} \Rightarrow G_2,\;\; F_1 \Rightarrow F_1, G_1 \Rightarrow G_1,\;\; F_3 \Rightarrow F_3, G_3 \Rightarrow G_3.$$

So the system (11) takes on the form

$$\left(\frac{d}{dr}+\frac{1-a}{2r}\right)F_3 + \frac{1+a}{r}F_2 + \frac{b}{2r}F_1 = 0,\;\; -2i\varepsilon G_2 - i\varepsilon G_3 - 2\left(\frac{d}{dr}-\frac{1+a}{2r}\right)G_0 + \frac{d}{dr}F_3 + \frac{1+a}{r}F_2 = 0,$$

$$-\left(\frac{d}{dr}+\frac{1+a}{2r}\right)G_3 + \frac{1-a}{r}G_2 + \frac{b}{2r}G_1 = 0,\;\; -2i\varepsilon F_2 + i\varepsilon F_3 - 2\left(\frac{d}{dr}-\frac{1-a}{2r}\right)F_0 - \frac{d}{dr}G_3 + \frac{1-a}{r}G_2 = 0, \quad (15)$$

$$-i\varepsilon F_1 + \frac{d}{dr}G_1 + \frac{b}{r}G_2 + \frac{b}{r}F_0 = 0,\quad -i\varepsilon G_1 + \frac{d}{dr}F_1 + \frac{b}{r}F_2 + \frac{b}{r}G_0 = 0,$$

$$-i\varepsilon F_3 + \frac{1+a}{r}F_0 + \frac{1+a}{2r}G_3 - \frac{b}{2r}G_1 = 0,\quad i\varepsilon G_3 + \frac{1-a}{r}G_0 - \frac{1-a}{2r}F_3 - \frac{b}{2r}F_1 = 0.$$

Let the first equation retain the same; from the first equation we subtract the second; the third equation retains the same; from the third equation subtract the fourth; remaining four equations retain the same. In this way we obtain

$$1)\; \left(\frac{d}{dr}+\frac{1-a}{2r}\right)F_3 + \frac{1+a}{r}F_2 + \frac{b}{2r}F_1 = 0, \qquad 2)\; 2\left(\frac{d}{dr}-\frac{1+a}{2r}\right)G_0 + \frac{1-a}{2r}F_3 + \frac{b}{2r}F_1 + 2i\varepsilon G_2 + i\varepsilon G_3 = 0,$$

$$3)\; -\left(\frac{d}{dr}+\frac{1+a}{2r}\right)G_3 + \frac{1-a}{r}G_2 + \frac{b}{2r}G_1 = 0, \qquad 4)\; 2\left(\frac{d}{dr}-\frac{1-a}{2r}\right)F_0 - \frac{1+a}{2r}G_3 + \frac{b}{2r}G_1 + 2i\varepsilon F_2 - i\varepsilon F_3 = 0,$$

$$5)\; -i\varepsilon F_1 + \frac{d}{dr}G_1 + \frac{b}{r}G_2 + \frac{b}{r}F_0 = 0, \qquad 6)\; -i\varepsilon G_1 + \frac{d}{dr}F_1 + \frac{b}{r}F_2 + \frac{b}{r}G_0 = 0,$$

$$7)\; -i\varepsilon F_3 + \frac{1+a}{r}F_0 + \frac{1+a}{2r}G_3 - \frac{b}{2r}G_1 = 0, \qquad 8)\; i\varepsilon G_3 + \frac{1-a}{r}G_0 - \frac{1-a}{2r}F_3 - \frac{b}{2r}F_1 = 0.$$

From 2) and 4) we can find expressions for F_2 and G_2:

$$F_2 = i\frac{(a-1)F_0 + (a-1)G_2 + r(-i\varepsilon F_3 + 2F_0 + G_3)}{2r\varepsilon}, G_2 = -i\frac{(a+1)G_0 + (a+1)F_2 + r(F_3 - i\varepsilon G_3 - 2G_0)}{2r\varepsilon}.$$

From 7) and 8) we can find expressions for F_3 and G_3:

$$F_3 = \frac{2(a^2-1)G_0 + (a+1)(bF_1 + 4ir\varepsilon F_0) - 2ibr\varepsilon G_1}{a^2-4r^2\varepsilon^2-1}, G_3 = \frac{-2(a^2-1)F_0 + (a-1)(bG_1 + 4ir\varepsilon G_0) + 2ibr\varepsilon F_1}{a^2-4r^2\varepsilon^2-1}.$$

Further we substitute these four formulas into 1), 3), 5), 6). In this way, we obtain two pairs of equations for variables F_0, F_1, G_0, G_1:

$$-\frac{2i(a^2-1)\varepsilon(3a^2-4r^2\varepsilon^2-3)}{(a^2-4r^2\varepsilon^2-1)^2}F_0+\frac{2(a^2-1)(-a^3+a+4r^2\varepsilon^2)}{r(a^2-4r^2\varepsilon^2-1)^2}G_0+$$
$$+\frac{2br\varepsilon^2(-3a^2+4r^2\varepsilon^2+3)}{(a^2-4r^2\varepsilon^2-1)^2}F_1+\frac{2i(a-1)(a^2-1)b\varepsilon}{(a^2-4r^2\varepsilon^2-1)^2}G_1+$$
$$+\frac{2(a^2-1)}{a^2-4r^2\varepsilon^2-1}G_0'+\frac{2ibr\varepsilon(-a^2+4r^2\varepsilon^2+1)}{(a^2-4r^2\varepsilon^2-1)^2}G_1'=0,$$

$$+\frac{4br\varepsilon^2(-3a^2+4r^2\varepsilon^2+3)}{(a^2-4r^2\varepsilon^2-1)^2}F_0-\frac{4ib\varepsilon(-a^3+a+4r^2\varepsilon^2)}{(a^2-4r^2\varepsilon^2-1)^2}G_0+$$
$$+\left(\frac{ib^2\varepsilon(a^2+4r^2\varepsilon^2-1)}{(a^2-4r^2\varepsilon^2-1)^2}-i\varepsilon\right)F_1+\frac{4(a-1)b^2r\varepsilon^2}{(a^2-4r^2\varepsilon^2-1)^2}G_1+$$
$$+\frac{4ibr\varepsilon(-a^2+4r^2\varepsilon^2+1)}{(a^2-4r^2\varepsilon^2-1)^2}G_0'+\left(\frac{b^2(-a^2+4r^2\varepsilon^2+1)}{(a^2-4r^2\varepsilon^2-1)^2}+1\right)G_1'=0;$$

and

$$\frac{2(a^2-1)(a^3-a+4r^2\varepsilon^2)}{r(a^2-4r^2\varepsilon^2-1)^2}F_0-\frac{2i(a^2-1)\varepsilon(3a^2-4r^2\varepsilon^2-3)}{(a^2-4r^2\varepsilon^2-1)^2}G_0-$$
$$-\frac{2i(a-1)(a+1)^2b\varepsilon}{(a^2-4r^2\varepsilon^2-1)^2}F_1+\frac{2br\varepsilon^2(-3a^2+4r^2\varepsilon^2+3)}{(a^2-4r^2\varepsilon^2-1)^2}G_1+\frac{2(a^2-1)}{a^2-4r^2\varepsilon^2-1}F_0'-\frac{2ibr\varepsilon}{a^2-4r^2\varepsilon^2-1}F_1'=0,$$

$$-\frac{4ib\varepsilon(a^3-a+4r^2\varepsilon^2)}{(a^2-4r^2\varepsilon^2-1)^2}F_0+\frac{4br\varepsilon^2(-3a^2+4r^2\varepsilon^2+3)}{(a^2-4r^2\varepsilon^2-1)^2}G_0-$$
$$-\frac{4(a+1)b^2r\varepsilon^2}{(a^2-4r^2\varepsilon^2-1)^2}F_1-\frac{i\varepsilon(-4r^2\varepsilon^2(2a^2+b^2-2)+(a^2-1)(a^2-b^2-1)+16r^4\varepsilon^4)}{(a^2-4r^2\varepsilon^2-1)^2}G_1-$$
$$-\frac{4ibr\varepsilon}{a^2-4r^2\varepsilon^2-1}F_0'+\frac{(-a^2+4r^2\varepsilon^2+1)(-a^2+b^2+4r^2\varepsilon^2+1)}{(a^2-4r^2\varepsilon^2-1)^2}F_1'=0.$$

These pairs may be considered as linear systems with respect to the variables F'_0, F'_1, and G'_0, G'_1. After simple calculations we find their solutions:

$$F_0'=\frac{ir\varepsilon(-3a^2+3+4r^2\varepsilon^2)G_0+(a^3-a+4r^2\varepsilon^2)F_0-i(1+a)br\varepsilon F_1-2br^2\varepsilon^2G_1}{-a^2r+4r^3\varepsilon^2+r}, \tag{16}$$

$$G_0'=\frac{ir\varepsilon(-3a^2+3+4r^2\varepsilon^2)F_0-(a^3-a-4r^2\varepsilon^2)G_0-2br^2\varepsilon^2F_1-i(1-a)br\varepsilon G_1}{-a^2r+4r^3\varepsilon^2+r}, \tag{17}$$

$$G_1'=i\varepsilon F_1, \qquad F_1'=i\varepsilon G_1. \tag{18}$$

From (18) follow separate equations for F_1 and G_1: $F_1''+\varepsilon^2F_1=0, G_1''+\varepsilon^2G_1=0$; their linearly independent solutions are

$$\mu=+1, \quad F_1^+=e^{+i\varepsilon r}\Rightarrow G_1^+=e^{+i\varepsilon r}; \ \mu=-1, \quad F_1^-=e^{-i\varepsilon r}\Rightarrow G_1^-=-e^{-i\varepsilon r}. \tag{19a}$$

Independent solutions may be chosen differently:

$$I,\ F_1^I = \cos \varepsilon r,\quad G_1^I = +i \sin \varepsilon r;\qquad II,\ F_1^{II} = \sin \varepsilon r,\quad G_1^{II} = -i \cos \varepsilon r. \tag{19b}$$

Solutions (19a) and (19b) are related by linear transformations:

$$F_1^{II} = \sin \varepsilon r = \frac{1}{2i}(F_1^+ - F_1^-),\ G_1^I = i \sin \varepsilon r = \frac{1}{2i}(G_1^+ - G_1^-),$$

$$F_1^I = \cos \varepsilon r = \frac{1}{2}(F_1^+ + F_1^-),\ G_1^{II} = -i \cos \varepsilon r = \frac{1}{2}(G_1^+ + G_1^-).$$

Applying the elimination method, from (16), (17) we derive equations (we follow both variants):

$$\mu = +1, \qquad -\frac{be^{ir\varepsilon}(-3(a-1)a + 2r\varepsilon(2r\varepsilon + 3i))}{-3a^2 + 4r^2\varepsilon^2 + 3} -$$
$$-\frac{8ir^2\varepsilon G_0'}{-3a^2 + 4r^2\varepsilon^2 + 3} + \frac{irG_0''}{\varepsilon} + \frac{iG_0((15 - 7a^2)r^2\varepsilon^2 + 3(a-1)^2 a(a+1) + 4r^4\varepsilon^4)}{r\varepsilon(-3a^2 + 4r^2\varepsilon^2 + 3)} = 0,$$

$$\frac{be^{ir\varepsilon}(3a(a+1) - 2r\varepsilon(2r\varepsilon + 3i))}{-3a^2 + 4r^2\varepsilon^2 + 3} -$$
$$-\frac{8ir^2\varepsilon F_0'}{-3a^2 + 4r^2\varepsilon^2 + 3} + \frac{irF_0''}{\varepsilon} + \frac{iF_0((15 - 7a^2)r^2\varepsilon^2 + 3(a-1)a(a+1)^2 + 4r^4\varepsilon^4)}{r\varepsilon(-3a^2 + 4r^2\varepsilon^2 + 3)} = 0; \tag{20}$$

$$\mu = -1, \qquad \frac{ib\varepsilon e^{-ir\varepsilon}(3(a-1)a + 2r\varepsilon(-2r\varepsilon + 3i))}{-3a^2 + 4r^2\varepsilon^2 + 3} -$$
$$-\frac{8r^2\varepsilon^2 G_0}{-3a^2 + 4r^2\varepsilon^2 + 3} + \frac{G_0((15 - 7a^2)r^2\varepsilon^2 + 3(a-1)^2 a(a+1) + 4r^4\varepsilon^4)}{r(-3a^2 + 4r^2\varepsilon^2 + 3)} + rG_0 = 0,$$

$$\frac{b\varepsilon e^{-ir\varepsilon}(2r\varepsilon(3 + 2ir\varepsilon) - 3ia(a+1))}{-3a^2 + 4r^2\varepsilon^2 + 3} -$$
$$-\frac{8r^2\varepsilon^2 F_0}{-3a^2 + 4r^2\varepsilon^2 + 3} + \frac{F_0((15 - 7a^2)r^2\varepsilon^2 + 3(a-1)a(a+1)^2 + 4r^4\varepsilon^4)}{r(-3a^2 + 4r^2\varepsilon^2 + 3)} + rF_0 = 0. \tag{21}$$

Equations (20), (21) permit us with the help relations

$$F_3 = \frac{2(a^2 - 1)G_0 + (a+1)(bF_1 + 4ir\varepsilon F_0) - 2ibr\varepsilon G_1}{a^2 - 4r^2\varepsilon^2 - 1},$$
$$G_3 = \frac{-2(a^2 - 1)F_0 + (a-1)(bG_1 + 4ir\varepsilon G_0) + 2ibr\varepsilon F_1}{a^2 - 4r^2\varepsilon^2 - 1};\quad F_2 = -G_0,\quad G_2 = -F_0 \tag{22}$$

to find concomitant functions $F_2, G_2; F_3, G_3$.

To solve general solutions of nonhomogeneous equations for F_0, G_0, we should have known particular solutions of them and general solutions of corresponding homogeneous equations. Particular solutions may be found through the use of the known gauge solutions. To this end, let us turn to gauge solutions (14), taking in mind the change in notations according to (15):

$$\bar{F}_0 = -i\varepsilon(K_1 + K_2)\frac{1}{\sqrt{2}},\ \bar{F}_1 = -\frac{b}{\sqrt{2}r}(K_1 + K_2),\ \bar{F}_2 = \left(\frac{d}{dr} - \frac{1}{r}\right)(K_1 + K_2)\frac{1}{\sqrt{2}},\ \bar{F}_3 = -\frac{a+1}{\sqrt{2}r}(K_1 + K_2),$$

$$G_0 = -i\varepsilon(K_1 - K_2)\frac{1}{\sqrt{2}},\ \bar{G}_1 = -\frac{b}{\sqrt{2}r}(K_1 - K_2),\ \bar{G}_2 = \left(\frac{d}{dr} - \frac{1}{r}\right)(K_1 - K_2)\frac{1}{\sqrt{2}},\ \bar{G}_3 = -\frac{a-1}{\sqrt{2}r}(K_1 - K_2).$$

Particular solutions should be determined as follows

$$\mu=+1, K_1=-\frac{r\sqrt{2}}{b}e^{+i\varepsilon r}, K_2=0; \qquad \mu=+1, K_1=0, K_2=-\frac{r\sqrt{2}}{b}e^{-i\varepsilon r}.$$

Thus we have fixed two gauge solutions:

$$\mu=+1, \qquad \bar{F}_0=\frac{i\varepsilon r}{b}e^{+i\varepsilon r},\ G_0=\frac{i\varepsilon r}{b}e^{+i\varepsilon r},\ \bar{F}_1=e^{+i\varepsilon r},\ \bar{G}_1=e^{+i\varepsilon r},$$

$$\bar{F}_2=-\left(\frac{d}{dr}-\frac{1}{r}\right)\frac{r}{b}e^{+i\varepsilon r},\ \bar{G}_2=-\left(\frac{d}{dr}-\frac{1}{r}\right)\frac{r}{b}e^{+i\varepsilon r},\ \bar{F}_3=\frac{a+1}{b}e^{+i\varepsilon r},\ \bar{G}_3=\frac{a-1}{b}e^{+i\varepsilon r}; \tag{23}$$

$$\mu=-1, \qquad \bar{F}_0=\frac{i\varepsilon r}{b}e^{-i\varepsilon r},\ G_0=-\frac{i\varepsilon r}{b}e^{-i\varepsilon r},\ \bar{F}_1=e^{-i\varepsilon r},\ \bar{G}_1=-e^{-i\varepsilon r},$$

$$\bar{F}_2=-\left(\frac{d}{dr}-\frac{1}{r}\right)\frac{r}{b}e^{-i\varepsilon r},\ \bar{G}_2=\left(\frac{d}{dr}-\frac{1}{r}\right)\frac{r}{b}e^{-i\varepsilon r},\ \bar{F}_3=\frac{a+1}{b}e^{-i\varepsilon r},\ \bar{G}_3=-\frac{a-1}{b}e^{-i\varepsilon r}. \tag{24}$$

By direct calculation we can verify that the functions F_0, G_0 from (23) and (24) indeed provide us with exact solutions of equations (20) and (21). Results are the same when applying solutions (19b). It should be emphasized that the doubling $\mu=+1$ and $\mu=-1$ (as well as I and II) refers to existence of linearly independent solutions of the 2nd order equation, but not to the degrees of freedom of the spin 3/2 particle. This method provides us with two solutions with opposite parities which do not contain gauge constituents.

Solving the homogeneous equations. Let us consider homogeneous equations from (21), (22):

$$\mu=+1, \qquad G_0''-\frac{8r\varepsilon^2}{(-3a^2+4r^2\varepsilon^2+3)}G_0'+\frac{(15-7a^2)r^2\varepsilon^2+3(a-1)^2a(a+1)+4r^4\varepsilon^4}{r^2(-3a^2+4r^2\varepsilon^2+3)}G_0=0,$$

$$F_0''-\frac{8r\varepsilon^2}{(-3a^2+4r^2\varepsilon^2+3)}F_0'+\frac{(15-7a^2)r^2\varepsilon^2+3(a-1)a(a+1)^2+4r^4\varepsilon^4}{r^2(-3a^2+4r^2\varepsilon^2+3)}F_0=0;$$

$$\mu=-1, \qquad G_0''-\frac{8r\varepsilon^2}{(-3a^2+4r^2\varepsilon^2+3)}G_0'+\frac{(15-7a^2)r^2\varepsilon^2+3(a-1)^2a(a+1)+4r^4\varepsilon^4}{r^2(-3a^2+4r^2\varepsilon^2+3)}G_0=0,$$

$$F_0''-\frac{8r\varepsilon^2}{(-3a^2+4r^2\varepsilon^2+3)}F_0'+\frac{(15-7a^2)r^2\varepsilon^2+3(a-1)a(a+1)^2+4r^4\varepsilon^4}{r^2(-3a^2+4r^2\varepsilon^2+3)}F_0=0.$$

All four equations contain one the same 2nd order operator, so it suffices to study only one equation:

$$\frac{d^2}{dr^2}f-\frac{8r\varepsilon^2}{(-3a^2+4r^2\varepsilon^2+3)}\frac{d}{dr}f+\frac{(15-7a^2)r^2\varepsilon^2+3(a-1)a(a+1)^2+4r^4\varepsilon^4}{r^2(-3a^2+4r^2\varepsilon^2+3)}f=0;$$

note identities $a^2-1=b^2$, $(a-1)a(a+1)^2=b^2a(a+1)$, $15-7a^2=8+7-7a^2=8-7b^2$.

It is convenient to use dimensionless variable $\varepsilon r=x$, so we get

$$\frac{d^2f}{dx^2}-\frac{8x}{(4x^2-3b^2)}\frac{df}{dx}+\frac{4x^4+(8-7b^2)x^2+3b^2a(a+1)}{x^2(4x^2-3b^2)}f=0.$$

The equation under consideration has three regular points and one irregular $x=\infty$ of the rank 2. Solutions are searched in the form $f(x)=x^Ae^{Bx}F(x)$. For function $F(x)$ we get the equation

$$\frac{d^2F}{dx^2}+\left(2B+\frac{2A}{x}-\frac{8x}{4x^2-3b^2}\right)\frac{dF}{dx}+$$

$$+\left[B^2+1+\frac{2AB}{x}+\frac{A(A-1)-a(a+1)}{x^2}+\frac{-8Bx-8A+4a(a+1)-4b^2+8}{4x^2-3b^2}\right]F=0.$$

At the choice $A=-a, a+1$, $B=\pm i$, the above equation simplifies

$$\frac{d^2F}{dx^2}+\left(2B+\frac{2A}{x}-\frac{8x}{4x^2-3b^2}\right)\frac{dF}{dx}+\left[\frac{2AB}{x}+\frac{-8Bx-8A+4a(a+1)-4b^2+8}{4x^2-3b^2}\right]F=0.$$

Functions $F(x)$ are constructed as power series with the 4-terms recurrent relations, convergence of the relevant series is studied: $R_{\text{conv}}=\sqrt{3}b/2, \infty$.

Conclusions. The system of equations for the massless spin 3/2 field has been studied in the spherical coordinates of Minkowski space. General structure of the spherical gauge solutions is specified, and it is demonstrated that the gauge radial functions satisfy the derived system of 8 equations. It is proved that the general system reduces to two couples of independent 2nd order and nonhomogeneous differential equations, their particular solutions may be found with the use of the gauge solutions. The corresponding homogeneous equations turn out to have one the same form, and have three regular singularities and one irregular of the rank 2. Frobenius types solutions for this equation have been constructed. Six remaining radial functions may be straightforwardly found though the use of the simple algebraic relations. This method provides us with two solutions with opposite parities which do not contain gauge constituents.

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