ISSN 1561-8323 (Print)
ISSN 2524-2431 (Online)

ФИЗИКА

PHYSICS

UDC 539.12
https://doi.org/10.29235/1561-8323-2024-68-1-18-27

Received 07.09.2023
Поступило в редакцию 07.09.2023

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NON-RELATIVISTIC APPROXIMATION IN THE PAULI–FIERZ THEORY FOR A SPIN 3/2 PARTICLE IN THE PRESENCE OF EXTERNAL FIELDS

(Communicated by Corresponding Member Dmitry S. Mogilevtsev)

Abstract. In the paper, we examine the non-relativistic approximation in the relativistic system of equations in Cartesian coordinates for 16-component wave functions with transformation properties of the vector-bispinor under the Lorentz group. When performing the non-relativistic approximation, for separating large and small components in the complete wave function we apply the method of projective operators. Accordingly, the complete wave function is presented as a sum of three parts: the large part depends on 6 variables, and the small ones depend on 14 variables. We have found two linear constraints on large components and two constraints on the small ones. After performing the procedure of the non-relativistic approximation we have derived 6 equations with a needed non-relativistic structure, which include only 4 large components. It is proved that only 4 equations are independent, so we have arrived at the generalized Pauli-like equation for the 4-component wave function. The analysis of transformation properties of the non-relativistic wave function permits us to generalize the structure of the derived equation to an arbitrary curved 3-space.

Keywords: spin 3/2 particle, external electromagnetic field, Cartesian coordinates, nonrelativistic approximation, projective operators, curved 3-space, tetrad formalism

For citation. Ivashkevich A. V., Red'kov V. M., Ishkhanyan A. M. Non-relativistic approximation in the Pauli–Fierz theory for a spin 3/2 particle in the presence of external fields. *Doklady Natsional'noi akademii nauk Belarusi = Doklady of the National Academy of Sciences of Belarus*, 2024, vol. 68, no. 1, pp. 18–27. https://doi.org/10.29235/1561-8323-2024-68-1-18-27

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НЕРЕЛЯТИВИСТСКОЕ ПРИБЛИЖЕНИЕ В ТЕОРИИ ПАУЛИ–ФИРЦА ДЛЯ ЧАСТИЦЫ СО СПИНОМ 3/2 В ПРИСУТСТВИИ ВНЕШНИХ ПОЛЕЙ

(Представлено членом-корреспондентом Д. С. Могилевцевым)

Аннотация. Исследуется нерелятивистское приближение для релятивистской системы из 16 уравнений в декартовых координатах для волновой функции частицы со спином 3/2 с трансформационными свойствами вектор-биспинора относительно группы Лоренца. При осуществлении нерелятивистского приближения для выделения в волновой функции больших и малых составляющих используется метод проективных операторов. Соответственно полная волновая функция представляется в виде суммы трех частей: зависящей от 6 переменных большой составляющей и двух малых составляющих, зависящих в совокупности от 14 переменных. Найдены два линейных ограничения на 6 больших компонент, и 2 ограничения на 14 малых. После выполнения процедуры нерелятивистского приближения выведено 6 уравнений с нерелятивистской структурой относительно 4 больших компонент. Показано, что только 4 уравнения из 6 являются независимыми. В результате найдено уравнение паулиевского типа для 4-компонентной волновой функции. Найден явный вид трех 4-мерных матриц, представляющих спин частицы. Анализ трасформационных свойств нерелятивистской волновой функции позволяет обобщить структуру найденного уравнения на случай искривленного трехмерного пространства.

Ключевые слова: частица со спином 3/2, внешнее электромагнитное поле, декартовые координаты, нерелятивистское приближение, проективные операторы, тетрадный формализм, искривленное трехмерное пространство

Для цитирования. Ивашкевич, А. В. Нерелятивистское приближение в теории Паули–Фирца для частицы со спином 3/2 в присутствии внешних полей / А. В. Ивашкевич, В. М. Редьков, А. М. Ишханян // Докл. Нац. акад. наук Беларуси. – 2024. – Т. 68, № 1. – С. 18–27. https://doi.org/10.29235/1561-8323-2024-68-1-18-27

Introduction. It is evident that non-relativistic equations are solved easier than relativistic ones. In the present paper we derive the non-relativistic equation for spin 3/2 particle in presence of external electromagnetic fields.

We start with the relativistic system of equations for 16-component wave functions with transformation properties of vector-bispinor under the lorentz group [1–9]. When performing the non-relativistic approximation, for separating in the complete wave function big and small components we apply the method of projective operators. Correspondingly, the complete wave function is presented as a sum of three parts: the big Ψ_+, depending on 6 variables, and the small Ψ_0 and Ψ_-, depending on 14 variables. There are found 2 linear constraints on big components, and 2 constraints on the small ones. The system of equations is presented in explicit form with the use of 20 new variables. After performing the procedure of the non-relativistic approximation we derive 6 equations with the needed non-relativistic structure, in which enter only 4 main primary big components. It is proved that only 4 equations are independent, so we arrive at the generalized Pauli-like equation for 4-component wave function.

Initial covariant equation. Let us start with the tetrad based form of the master equation for spin 3/2 particle [10–15]

$$\gamma^5 \varepsilon_k^{\ can} \gamma_c \left[i(D_a)_n^{\ l} - \frac{1}{2} M \gamma_a \delta_n^{\ l} \right] \Psi_l = 0, \tag{1}$$

where $M = mc / \hbar$ is a mass parameter, the presence of the multiplier is meaningful; the generalized derivative are determined by the formula

$$D_a = e^{\alpha}_{(a)} (\partial_\alpha + ieA_\alpha) + \frac{1}{2} (\sigma^{ps} \otimes I + I \otimes j^{ps}) \gamma_{[ps]a}.$$

With the use of six matrices $\varepsilon_k^{\ can} = (\mu^{[ca]})_k^{\ n}$, eq. (1) may be presented as follows

$$\gamma^5 (\mu^{[ca]})_k^{\ n} \gamma_c \left[i(D_a)_n^{\ l} - M \gamma_a \delta_n^{\ l} \right] \Psi_l = 0,$$

whence we derive the detailed form of eq. (1):

$$\left(\gamma^1 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[03]}\right) D_0 \Psi + \left(\gamma^0 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[12]} - \gamma^3 \otimes \mu^{[31]}\right) D_1 \Psi +$$
$$+ \left(\gamma^0 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[23]} - \gamma^1 \otimes \mu^{[12]}\right) D_2 \Psi + \left(\gamma^0 \otimes \mu^{[03]} + \gamma^1 \otimes \mu^{[31]} - \gamma^2 \otimes \mu^{[23]}\right) D_3 \Psi +$$
$$+ iM \frac{1}{2} \{ s_{01} \otimes \mu^{[01]} + s_{02} \otimes \mu^{[02]} + s_{03} \otimes \mu^{[03]} + s_{23} \otimes \mu^{[23]} +$$
$$+ s_{31} \otimes \mu^{[31]} + s_{12} \otimes \mu^{[12]} \} \Psi = 0, \ \ s_{ab} = \gamma_a \gamma_b - \gamma_b \gamma_a.$$

The above equation may be presented shortly as follows

$$(\Gamma^0 D_0 + \Gamma^0 D_0 + \Gamma^0 D_0 + \Gamma^0 D_0 + iM\Gamma) \Psi = 0. \tag{2}$$

It is convenient to multiply eq. (2) by the matrix Γ^{-1}, so we get

$$(Y^0 D_0 + Y^1 D_1 + Y^2 D_2 + Y^3 D_3 + iM) \Psi = 0. \tag{3}$$

Nonrelativistic approximation. We restrict ourselves to Minkowski space-time model and Cartesian coordinate. The wave function may be presented in the matrix form

$$\Psi_{A(n)} = \begin{vmatrix} f_0 & f_1 & f_2 & f_3 \\ g_0 & g_1 & g_2 & g_3 \\ h_0 & h_1 & h_2 & h_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix}, \ \ \Psi = \{ f_0, g_0, h_0, d_0; f_1, g_1, h_1, d_1; f_2, g_2, h_2, d_2; f_3, g_3, h_3, d_3 \};$$

we calculate the term

$$\Gamma^0\Psi = \gamma^1\Psi\tilde{\mu}^{[01]} + \gamma^2\Psi\tilde{\mu}^{[02]} + \gamma^3\Psi\tilde{\mu}^{[03]} = \begin{vmatrix} 0 & id_3+h_2 & d_3-h_1 & -id_1-d_2 \\ 0 & -d_2-ih_3 & d_1+h_3 & ih_1-h_2 \\ 0 & -f_2-ig_3 & f_1-g_3 & ig_1+g_2 \\ 0 & if_3+g_2 & -f_3-g_1 & f_2-if_1 \end{vmatrix},$$

we derive its 16-dimensional representation, the same is done for all other matrices. We can readily prove that the minimal equation for the matrix for $Y^0 = Y_0$ is $Y_0^2(Y_0^2-1)=0$. So, we can define three projective operators $P_0 = 1-Y_0^2$, $P_1 = P_+ = +\frac{1}{2}Y_0^2(Y+1)$, $P_2 = P_- = -\frac{1}{2}Y_0^2(Y-1)$.

Presentation for three projective constituents is

$$\Psi_0 = \begin{vmatrix} f_0 \\ g_0 \\ h_0 \\ d_0 \\ (f_1+if_2-g_3)/3 \\ (f_3+g_1-ig_2)/3 \\ (-d_3+h_1+ih_2)/3 \\ (d_1-id_2+h_3)/3 \\ (-if_1+f_2+ig_3)/3 \\ i(f_3+g_1-ig_2)/3 \\ (i(d_3-h_1)+h_2)/3 \\ i(d_1-id_2+h_3)/3 \\ (f_3+g_1-ig_2)/3 \\ (-f_1-if_2+g_3)/3 \\ (d_1-id_2+h_3)/3 \\ (d_3-h_1-ih_2)/3 \end{vmatrix} = \begin{vmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ -iS_5 \\ iS_6 \\ -iS_7 \\ iS_8 \\ S_6 \\ -S_5 \\ S_8 \\ -S_7 \end{vmatrix}, \quad \Psi_+ = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (d_3+2f_1-if_2+g_3+2h_1-ih_2)/6 \\ (2d_1+id_2-f_3+2g_1+ig_2-h_3)/6 \\ (d_3+2f_1-if_2+g_3+2h_1-ih_2)/6 \\ (2d_1+id_2-f_3+2g_1+ig_2-h_3)/6 \\ -i(d_3-f_1+2if_2+g_3-h_1+2ih_2)/6 \\ -i(d_1+2id_2+f_3+g_1+2ig_2+h_3)/6 \\ -i(d_3-f_1+2if_2+g_3-h_1+2ih_2)/6 \\ -i(d_1+2id_2+f_3+g_1+2ig_2+h_3)/6 \\ (-d_1+id_2+2f_3-g_1+ig_2+2h_3)/6 \\ (2d_3+f_1+if_2+2g_3+h_1+ih_2)/6 \\ (-d_1+id_2+2f_3-g_1+ig_2+2h_3)/6 \\ (2d_3+f_1+if_2+2g_3+h_1+ih_2)/6 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ L_1 \\ L_2 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_5 \\ L_6 \end{vmatrix},$$

$$\Psi_- = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (-d_3+2f_1-if_2+g_3-2h_1+ih_2)/6 \\ (-2d_1-id_2-f_3+2g_1+ig_2+h_3)/6 \\ (d_3-2f_1+if_2-g_3+2h_1-ih_2)/6 \\ (2d_1+id_2+f_3-2g_1-ig_2-h_3)/6 \\ i(d_3+f_1-2if_2-g_3-h_1+2ih_2)/6 \\ i(d_1+2id_2-f_3-g_1-2ig_2+h_3)/6 \\ -i(d_3+f_1-2if_2-g_3-h_1+2ih_2)/6 \\ -i(d_1+2id_2-f_3-g_1-2ig_2+h_3)/6 \\ (d_1-id_2+2f_3-g_1+ig_2-2h_3)/6 \\ (-2d_3+f_1+if_2+2g_3-h_1-ih_2)/6 \\ (-d_1+id_2-2f_3+g_1-ig_2+2h_3)/6 \\ (2d_3-f_1-if_2-2g_3+h_1+ih_2)/6 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P_1 \\ P_2 \\ -P_1 \\ -P_2 \\ P_3 \\ P_4 \\ -P_3 \\ -P_4 \\ P_5 \\ P_6 \\ -P_5 \\ -P_6 \end{vmatrix}.$$

We should consider the Ψ_+ as large, whereas Ψ_- and Ψ_0 should be considered as small; projective constituents consist of the following variables

$$\Psi_+,\ \{L_1,...,L_6\};\quad \Psi_0,\ \{S_1,...,S_8\};\quad \Psi_-,\ \{P_1,...,P_6\}.$$

Constraints on large and small components. Let us consider relations which define the big variable

$$L_1=\frac{1}{6}(d_3+2f_1-if_2+g_3+2h_1-ih_2),$$

$$L_3=-\frac{1}{6}i(d_3-f_1+2if_2+g_3-h_1+2ih_2),$$

$$L_6=\frac{1}{6}(2d_3+f_1+if_2+2g_3+h_1+ih_2),$$

whence it follows the constraint $L_1+iL_3-L_6=0;$ and

$$L_2=\frac{1}{6}(2d_1+id_2-f_3+2g_1+ig_2-h_3),$$

$$L_4=-\frac{1}{6}i(d_1+2id_2+f_3+g_1+2ig_2+h_3),$$

$$L_5=\frac{1}{6}(-d_1+id_2+2f_3-g_1+ig_2+2h_3),$$

whence it follows $L_2-iL_4+L_5=0.$ Therefore, there exist only 4 independent ones:

$$iL_3=L_6-L_1,\quad iL_4=L_5+L_2. \tag{4}$$

Now let us consider relations which determine the constituent Ψ_-. Combing the relevant rows, we derive two identities

$$(A)\quad P_1+iP_3-P_6=0,\qquad (B)\quad P_2-iP_4+P_5=0;$$

they provide us with two constraints which will be used below. Now, let us consider relations which determine the sum of two small constituents

$$\Psi_0+\Psi_-=\begin{vmatrix} S_5+P_1\\ S_6+P_2\\ S_7-P_1\\ S_8-P_2\\ iS_5+P_3\\ iS_6+P_4\\ iS_7-P_3\\ iS_8-P_4\\ S_6+P_5\\ -S_5+P_6\\ S_8-P_5\\ -S_7-P_6\end{vmatrix}=\begin{vmatrix}+y_1\\+y_2\\+y_3\\+y_4\\+y_5\\+y_6\\+y_7\\+y_8\\+y_9\\+y_{10}\\+y_{11}\\+y_{12}\end{vmatrix}.$$

From these relations we can derive

$$y_1+y_3=S_5+S_7,\quad y_2+y_4=S_6+S_8,\quad y_5+y_7=i(S_5+S_7),$$

$$y_6+y_8=i(S_6+S_8),\quad y_9+y_{11}=S_6+S_8,\quad y_{10}+y_{12}=-(S_5+S_7);$$

and

$$(y_1+y_3)+(y_{10}+y_{12})=0,\quad (y_1+y_3)+i(y_5+y_7)=0,$$

$$(y_2+y_4)-(y_9+y_{11})=0, \quad (y_2+y_4)+i(y_6+y_8)=0;$$
$$S_5-S_7=(y_1-y_3)+i(y_5-y_7)-(y_{10}-y_{12}),$$
$$3(S_6-S_8)=(y_2-y_4)-i(y_6-y_8)+(y_9-y_{11}).$$

The study of the main system. Let us find 16 equations (3), using the presence of big and small variables, also taking into account the constrains (4). We omit their explicit form. Further we perform several steps in calculations: divide equations into 8 pairs; sum and subtract equations within each pair; when performing the non-relativistic approximation, we should take into account the separation of rest energy by the formal change

$$D_0 \Rightarrow (-iM+D_0);$$

also we should take into account the presence of small variables of different orders:

$$S_i \sim x, \quad y_s \sim x, \quad \frac{D_0}{M} \sim x^2, \quad \frac{D_j}{M} \sim x;$$

then we transform all equations to the new variables

$$y_1+y_3=\frac{1}{2}Z_1, \quad y_1-y_3=\frac{1}{2}Z_2, \quad y_2+y_4=\frac{1}{2}Z_3, \quad y_2-y_4=\frac{1}{2}Z_4,$$
$$y_5+y_7=\frac{1}{2}Z_5, \quad y_5-y_7=\frac{1}{2}Z_6, \quad y_6+y_8=\frac{1}{2}Z_7, \quad y_6-y_8=\frac{1}{2}Z_8,$$
$$y_9+y_{11}=\frac{1}{2}Z_9, \quad y_9-y_{11}=\frac{1}{2}Z_{10}, \quad y_{10}+y_{12}=\frac{1}{2}Z_{11}, \quad y_{10}-y_{12}=\frac{1}{2}Z_{12},$$

the six constraints are valid, but only 4 are independent:

$$Z_{11}=-Z_1, \quad iZ_5=-Z_1\ (Z_{11}=iZ_5); \quad Z_9=Z_3, \quad Z_7=iZ_3\ (Z_9=-iZ_7).$$

After that we can express all independent small components through the large ones:

$$\begin{vmatrix} X \\ Y \\ Z_2 \\ Z_4 \\ Z_6 \\ Z_8 \\ Z_{10} \\ Z_{12} \end{vmatrix} = \frac{1}{M} \begin{vmatrix} 2iD_1L_1+2iD_3L_5+2D_2(L_6-L_1) \\ 2iD_1L_2+2D_2(L_2+L_5)+2iD_3L_6 \\ -2iD_3L_1-2iD_1L_2-2D_2L_2 \\ -2iD_1L_1+2D_2L_1+2iD_3L_2 \\ -2D_1(L_2+L_5)+2iD_2(L_2+L_5)+2D_3(L_1-L_6) \\ 2D_3(L_2+L_5)+2D_1(L_1-L_6)+2iD_2(L_1-L_6) \\ -2iD_3L_5-2iD_1L_6-2D_2L_6 \\ -2iD_1L_5+2D_2L_5+2iD_3L_6 \end{vmatrix}$$

and then substitute these relations into the reaming equations.

In this way we arrive at 6 equations with non-relativistic structure, only 4 of 6 are independent. Thus we find four equations which contain only the 4 large components L_1, L_2, L_5, L_6. These 4 independent equations are transformed to the new variables

$$\Psi = \begin{vmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} L_1 \\ L_2 \\ L_5 \\ L_6 \end{vmatrix};$$

the final 4-component equation is presented in the form

$$iD_0\Psi = -\frac{1}{2M}\Delta\Psi + \frac{e}{3M}(S_1F_{23}+S_2F_{31}+S_3F_{12})\Psi;$$

where $D_0=\partial_0+ieA_0$, $\Delta=(\partial_1+ieA_1)^2+(\partial_2+ieA_2)^2+(\partial_3+ieA_3)^2$. Three matrices S_i obey the $SU(2)$ algebra, they may be considered as the spin matrices. There exists a basis in which the matrix S_3 becomes diagonal:

$$\bar{S}_1 = \begin{vmatrix} 0 & -1/2 & 0 & 0 \\ -3/2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -3/2 \\ 0 & 0 & -1/2 & 0 \end{vmatrix}, \quad \bar{S}_2 = \begin{vmatrix} 0 & -i/2 & 0 & 0 \\ 3i/2 & 0 & -i & 0 \\ 0 & i & 0 & -3i/2 \\ 0 & 0 & i/2 & 0 \end{vmatrix},$$

$$\bar{S}_3 = \begin{vmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 3/2 \end{vmatrix}.$$

Nonrelativistic approximation in curved 3-space. Nonrelativistic approximation (irrespective of the spin value of the particle; see in [10]) in space-times with the following metric

$$dS^2 = (dx^0)^2 + g_{ij}(x)dx^i dx^j, \quad e_{(a)\alpha}(x) = \begin{vmatrix} 1 & 0 \\ 0 & e_{(i)k}(x) \end{vmatrix}. \tag{5}$$

In such models expressions for connections become simpler [10]

$$\Gamma_0 = \frac{1}{2} J^{ik} e^m_{(i)} (\nabla_0 e_{(k)m}), \qquad \Gamma_l = \frac{1}{2} J^{ik} e^m_{(i)} (\nabla_l e_{(k)m}).$$

The contribution of J^{0k} vanishes; we apply the notation $J^{ik} = \sigma^{ik} \otimes I + I \otimes j^{ik}$. Let us derive a generally covariant nonrelativistic equation for the spin 3/2 particle in an arbitrary space with the structure (5).

To this end, we turn to transformation properties of the nonrelativistic wave function under rotation group

$$\Psi \sim \begin{vmatrix} \Phi_{1(0)} & \Phi_{1(1)} & \Phi_{1(2)} & \Phi_{1(3)} \\ \Phi_{2(0)} & \Phi_{2(1)} & \Phi_{2(2)} & \Phi_{2(3)} \\ \varphi_{1(0)} & F_{1(1)} & F_{1(2)} & F_{1(3)} \\ \varphi_{2(0)} & F_{2(1)} & F_{2(2)} & F_{2(3)} \end{vmatrix}, \quad \Psi_+ = \begin{vmatrix} 0 & L_1 & L_3 & L_5 \\ 0 & L_2 & L_4 & L_6 \\ 0 & L_1 & L_3 & L_5 \\ 0 & L_2 & L_4 & L_6 \end{vmatrix} =$$

$$= \begin{vmatrix} \Phi_{1(1)} & \Phi_{1(2)} & \Phi_{1(3)} \\ \Phi_{2(1)} & \Phi_{2(2)} & \Phi_{2(3)} \\ F_{1(1)} & F_{1(2)} & F_{1(3)} \\ F_{2(1)} & F_{2(2)} & F_{2(3)} \end{vmatrix} = \begin{vmatrix} \Phi \\ F \end{vmatrix}, \quad \Phi' = (B \otimes O)\Phi, \quad F' = (B \otimes O)F, \tag{6}$$

where the matrices B and O describe 3-rotations for 2-spinors and 3-vectors. It suffices to follow only two first rows. Let us find expressions for generators related to formulas (6):

$$J_j = (i/2)\sigma_j \otimes I + I \otimes V_j, \quad j = 1, 2, 3.$$

We readily find 6-dimensional representation for generators

$$\Phi = \begin{vmatrix} \Phi_{11} \\ \Phi_{12} \\ \Phi_{13} \\ \Phi_{21} \\ \Phi_{22} \\ \Phi_{23} \end{vmatrix} = \begin{vmatrix} L_1 \\ L_3 \\ L_5 \\ L_2 \\ L_4 \\ L_6 \end{vmatrix}, \quad J_1 = \begin{vmatrix} 0 & 0 & 0 & i/2 & 0 & 0 \\ 0 & 0 & -1 & 0 & i/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & i/2 \\ i/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & i/2 & 0 & 0 & 0 & -1 \\ 0 & 0 & i/2 & 0 & 1 & 0 \end{vmatrix},$$

$$J_2 = \begin{vmatrix} 0 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1/2 \\ -1/2 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & -1 & 0 & 0 \end{vmatrix}, \quad J_3 = \begin{vmatrix} i/2 & -1 & 0 & 0 & 0 & 0 \\ 1 & i/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & i/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i/2 \end{vmatrix}.$$

We find commutators

$$J_1J_2 - J_2J_1 = J_3 + K_3,\; J_2J_3 - J_3J_2 = J_1 + K_1,\; J_3J_1 - J_1J_3 = J_2 + K_2,$$

where

$$K_3 = \begin{vmatrix} -i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \end{vmatrix},\; K_1 = \begin{vmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{vmatrix},\; K_2 = \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix};$$

also we find $K_3^2 = -I, K_2^2 = -I, K_1^2 = -I,$ and

$$K_2K_3 + K_3K_2 = 0,\; K_3K_1 + K_1K_3 = 0,\; K_1K_2 + K_2K_1 = 0,$$

$$K_2K_3 = K_1,\; K_3K_1 = K_2,\; K_1K_2 = K_3.$$

The commutation relations

$$K_2K_3 - K_3K_2 = 2K_1,\; K_3K_1 - K_1K_3 = 2K_2,\; K_1K_2 - K_2K_1 = 2K_3$$

after transforming $S_i = \frac{1}{2}K_i$ take the form of the Lie algebra SO(3):

$$S_2S_3 - S_3S_2 = S_1,\; S_3S_1 - S_1S_3 = S_2,\; S_1S_2 - S_2S_1 = S_3. \tag{7}$$

Allowing for (7), we readily obtain the identity

$$(K_1D_1 + K_2D_2 + K_3D_3)^2 = -(D_1^2 + D_2^2 + D_3^2) +$$
$$+K_2K_3(D_2D_3 - D_3D_2) + K_3K_1(D_3D_1 - D_1D_3) + K_1K_2(D_1D_2 - D_2D_1),$$

whence it follows

$$\frac{1}{2M}(K_1D_1 + K_2D_2 + K_3D_3)^2 = -\frac{1}{2M}(D_1^2 + D_2^2 + D_3^2) + \frac{ie}{M}(F_{23}S_1 + F_{31}S_2 + F_{12}S_3), \tag{8}$$

which coincides with the structure of the nonrelativistic Hamiltonian in 6-demensional form. We can prove that (8) indeed leads to the above nonrelativistic equation for the spin 3/2 particle.

To this end, let us start with the explicit form of equation (8), whence we obtain (let $ie / 2M = \mu$)

$$iD_0L_1 = -\frac{1}{2M}D^2L_1 + \mu F_{23}(-iL_2) + \mu F_{31}(-L_2) + \mu F_{12}(-iL_1),$$

$$iD_0L_3 = -\frac{1}{2M}D^2L_3 + \mu F_{23}(-iL_4) + \mu F_{31}(-L_4) + \mu F_{12}(-iL_3),$$

$$iD_0L_5 = -\frac{1}{2M}D^2L_5 + \mu F_{23}(-iL_6) + \mu F_{31}(-L_6) + \mu F_{12}(-iL_5),$$

$$iD_0L_2 = -\frac{1}{2M}D^2L_2 + \mu F_{23}(-iL_1) + \mu F_{31}(+L_1) + \mu F_{12}(+iL_2),$$

$$iD_0L_4 = -\frac{1}{2M}D^2L_4 + \mu F_{23}(-iL_3) + \mu F_{31}(+L_3) + \mu F_{12}(+iL_4),$$

$$iD_0L_6 = -\frac{1}{2M}D^2L_6 + \mu F_{23}(-iL_5) + \mu F_{31}(+L_5) + \mu F_{12}(+iL_6).$$

Let us take into account two constraints $L_3 = (iL_1 - iL_6), L_4 = (-iL_2 - iL_5)$. This leads to

$$iD_0L_1 = -\frac{1}{2M}D^2L_1 + \mu F_{23}(-iL_2) + \mu F_{31}(-L_2) + \mu F_{12}(-iL_1),$$

$$iD_0(L_1 - L_6) = -\frac{1}{2M}D^2(L_1 - L_6) + \mu F_{23}(iL_2 + iL_5) + \mu F_{31}(L_2 + L_5) + \mu F_{12}(-iL_1 + iL_6),$$

$$iD_0L_5 = -\frac{1}{2M}D^2L_5 + \mu F_{23}(-iL_6) + \mu F_{31}(-L_6) + \mu F_{12}(-iL_5),$$

$$iD_0L_2 = -\frac{1}{2M}D^2L_2 + \mu F_{23}(-iL_1) + \mu F_{31}(+L_1) + \mu F_{12}(+iL_2),$$

$$iD_0(L_2 + L_5) = -\frac{1}{2M}D^2(L_2 + L_5) + \mu F_{23}(iL_1 - iL_6) + \mu F_{31}(+iL_3) + \mu F_{12}(iL_2 + iL_5),$$

$$iD_0L_6 = -\frac{1}{2M}D^2L_6 + \mu F_{23}(-iL_5) + \mu F_{31}(+L_5) + \mu F_{12}(+iL_6).$$

Let us divide these equations into two groups. Equations from the first group

$$iD_0L_1 = -\frac{1}{2M}D^2L_1 + \mu F_{23}(-iL_2) + \mu F_{31}(-L_2) + \mu F_{12}(-iL_1),$$

$$iD_0(L_1 - L_6) = -\frac{1}{2M}D^2(L_1 - L_6) + \mu F_{23}(iL_2 + iL_5) + \mu F_{31}(L_2 + L_5) + \mu F_{12}(-iL_1 + iL_6),$$

$$iD_0L_6 = -\frac{1}{2M}D^2L_6 + \mu F_{23}(-iL_5) + \mu F_{31}(+L_5) + \mu F_{12}(+iL_6)$$

we combine so that to get in the left side the variables $L_1 + L_6, 2L_1 - L_6, 2L_6 - L_1$. This results in

$$iD_0(L_1 + L_6) = -\frac{1}{2M}D^2(L_1 + L_6) + \mu F_{23}(-iL_2 - iL_5) + \mu F_{31}(-L_2 + L_5) + \mu F_{12}(-iL_1 + iL_6),$$

$$iD_0(2L_1 - L_6) = -\frac{1}{2M}D^2(2L_1 - L_6) + \mu F_{23}(-2iL_2 + iL_5) + \mu F_{31}(-2L_2 - L_5) + \mu F_{12}(-2iL_1 - iL_6),$$

$$iD_0(2L_6 - L_1) = -\frac{1}{2M}D^2(2L_6 - L_1) + \mu F_{23}(-2iL_5 + iL_2) + \mu F_{31}(2L_5 + L_2) + \mu F_{12}(2iL_6 + iL_1).$$

We can verify that the third equation is equal to the difference between the first and the second ones. Therefore, the third equation may be removed.

Equations from the second group may be studied in the same way: also there exist only two independent equations. Thus, we have only 4 independent equations

$$iD_0(L_1 + L_6) = -\frac{1}{2M}D^2(L_1 + L_6) + \mu F_{23}(-iL_2 - iL_5) + \mu F_{31}(-L_2 + L_5) + \mu F_{12}(-iL_1 + iL_6),$$

$$iD_0(L_2 - L_5) = -\frac{1}{2M}D^2(L_2 - L_5) + \mu F_{23}(-iL_1 + iL_6) + \mu F_{31}(L_1 + L_6) + \mu F_{12}(iL_2 + iL_5),$$

$$iD_0(2L_1 - L_6) = -\frac{1}{2M}D^2(2L_1 - L_6) + \mu F_{23}(-2iL_2 + iL_5) + \mu F_{31}(-2L_2 - L_5) + \mu F_{12}(-2iL_1 - iL_6),$$

$$iD_0(2L_2 + L_5) = -\frac{1}{2M}D^2(2L_2 + L_5) + \mu F_{23}(-2iL_1 - iL_6) + \mu F_{31}(2L_1 - L_6) + \mu F_{12}(2iL_2 - iL_5).$$

Let us introduce the new 4-component wave function

$$\Psi = \begin{vmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{vmatrix} = \begin{vmatrix}1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0\\ 2 & 0 & 0 & -1\\ 0 & 2 & 1 & 0\end{vmatrix}\begin{vmatrix}L_1\\ L_2\\ L_5\\ L_6\end{vmatrix}, \qquad \begin{array}{l} L_1 = \frac{1}{3}\Psi_1 + \frac{1}{3}\Psi_3, \quad L_2 = \frac{1}{3}\Psi_2 + \frac{1}{3}\Psi_4, \\ L_5 = \frac{1}{3}\Psi_4 - \frac{2}{3}\Psi_2, \quad L_6 = \frac{2}{3}\Psi_1 - \frac{1}{3}\Psi_3. \end{array}$$

Then the above system takes on the form

$$\begin{aligned}
&iD_0\Psi_1=-\frac{1}{2M}\Delta\Psi_1+\frac{ie}{M}\left(\frac{1}{6}iF_{23}(\Psi_2-2\Psi_4)-\frac{1}{2}F_{31}\Psi_2+\frac{1}{6}iF_{12}(\Psi_1-2\Psi_3)\right),\\
&iMD_0\Psi_2=-\frac{1}{2M}\Delta\Psi_2+\frac{ie}{M}\left(\frac{1}{6}iF_{23}(\Psi_1-2\Psi_3)+\frac{1}{2}F_{31}\Psi_1-\frac{1}{6}iF_{12}(\Psi_2-2\Psi_4)\right),\\
&iD_0\Psi_3=-\frac{1}{2M}\Delta\Psi_3+\frac{ie}{M}\left(\frac{1}{6}iF_{23}(\Psi_4-2\Psi_2)+\frac{1}{6}F_{31}(\Psi_4-2\Psi_2)-\frac{1}{2}iF_{12}\Psi_3\right),\\
&iMD_0\Psi_4=-\frac{1}{2}\Delta\Psi_4+ie\left(-\frac{1}{6}iF_{23}(2\Psi_1-\Psi_3)+\frac{1}{6}F_{31}(2\Psi_1-\Psi_3)+\frac{1}{2}iF_{12}\Psi_4\right).
\end{aligned} \tag{9}$$

The system (9) may be presented in the matrix form

$$iD_0\Psi=-\frac{1}{2M}\Delta\Psi+\frac{ie}{3M}(F_{23}S_1+F_{31}S_2+F_{12}S_3)\Psi,$$

$$S_1=\frac{i}{2}\begin{vmatrix}0&1&0&-2\\1&0&-2&0\\0&-2&0&1\\-2&0&1&0\end{vmatrix},\quad S_2=\frac{1}{2}\begin{vmatrix}0&-3&0&0\\3&0&0&0\\0&-2&0&1\\2&0&-1&0\end{vmatrix},\quad S_3=\frac{i}{2}\begin{vmatrix}1&0&-2&0\\0&-1&0&2\\0&0&-3&0\\0&0&0&3\end{vmatrix}.$$

The last matrices obey the commutation rule $S_1S_2-S_2S_1=S_3$, and so on; therefore, they may be considered as the components of the spin operator. We can readily find a transformation which makes the matrix S_3 diagonal

$$S_3\Psi=\sigma\Psi,\ \bar{\Psi}=S\Psi,\ S^{-1}\bar{\Psi}=\Psi,\ SS_3S^{-1}=\bar{S}_3.$$

The needed transformation is

$$S=\begin{vmatrix}0&0&0&1\\-2&0&1&0\\0&-2&0&1\\0&0&1&0\end{vmatrix},\quad S^{-1}=\begin{vmatrix}0&-1/2&0&1/2\\1/2&0&-1/2&0\\0&0&0&1\\1&0&0&0\end{vmatrix}.$$

In this new representation we have the following spin components

$$\bar{S}_1=\begin{vmatrix}0&-1/2&0&0\\-3/2&0&-1&0\\0&-1&0&-3/2\\0&0&-1/2&0\end{vmatrix},\quad \bar{S}_2=\begin{vmatrix}0&-i/2&0&0\\3i/2&0&-i&0\\0&i&0&-3i/2\\0&0&i/2&0\end{vmatrix},$$

$$\bar{S}_3=\begin{vmatrix}-3/2&0&0&0\\0&-1/2&0&0\\0&0&1/2&0\\0&0&0&3/2\end{vmatrix}.$$

Now we can easily generalize the above nonrelativistic equation to the generally covariant form. The structure of that equation should be as follows (we start with the 6-dimensional form)

$$iD_0\Psi_6=\frac{1}{2M}\left[K^j(x)\left(\frac{\partial}{\partial x^j}+\Gamma_j(x)+ieA_j(x)\right)\right]^2\Psi_6,$$

$$K^j(x)=K_ie^j_{(i)}(x),\quad \Gamma_j(x)=\frac{1}{2}J^{kl}e^n_{(k)}(x)\nabla_je_{(l)n}(x),$$

where the generalized derivative are determined by the formulas

$$D_0(x)=\partial_0+ieA_0(x)+\frac{1}{2}(\sigma^{ps}\otimes I+I\otimes j^{ps})\gamma_{[ps]0}(x),$$

$$D_k(x) = e^j_{(k)}(x)(\partial_j + ieA_j(x)) + \frac{1}{2}(\sigma^{ps} \otimes I + I \otimes j^{ps})\gamma_{[ps]k}(x), k = 1, 2, 3. \quad (10)$$

The definition of the 6 large components remains the same. As well as two linear constraints preserve their form. All algebraic transformations proving existence of only 4 independent equations also are the same. The difference consists only in the new and more complicated expressions for generalized derivatives. Correspondingly, we obtain the generalized equation

$$iD_0(x)\Psi = -\frac{1}{2M}(D_1^2(x) + D_2^2(x) + D_3^2(x))\Psi + \frac{1}{2M}(D_{[23]}S_1 + D_{[31]}S_2 + D_{[12]}S_3)\Psi_4, \quad (11)$$

where the commutators $D_{[kl]} = D_k(x)D_l(x) - D_l(x)D_k(x)$ are used.

Conclusion. The structure of the generalized Pauli-like equation (11) indicates that due to the presence of the covariant derivatives (10) the final explicit form of equation (11) will include in addition to electromagnetic interaction also the geometrical interaction term through the Ricci tensor,

Acknowledgments. The work was carried out with financial support from the Armenian Science Committee (grant no. 21AG-1C064 and 21SC-BRFFR-1C021) and the Belarusian Republican Foundation for Fundamental Research (project Ф21Арм-022).

Благодарности. Работа выполнена при финансовой поддержке Комитета науки Армении (гранты № 21AG-1C064 и 21SC-BRFFR-1C021) и Белорусского республиканского фонда фундаментальных исследований (проект Ф21Арм-022).

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