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CLASSICAL SOLUTION OF A MIXED PROBLEM FOR THE WAVE EQUATION WITH DISCONTINUOUS INITIAL CONDITIONS IN A CURVILINEAR HALF-STRIP

Abstract. For a one-dimensional wave equation, we consider a mixed problem in a curvilinear half-strip. The initial conditions have a first-kind discontinuity at one point. The mixed problem models the problem of a longitudinal impact on a finite elastic rod with a movable boundary. We construct the solution using the method of characteristics in an explicit analytical form. For the problem in question, we prove the uniqueness of the solution and establish the conditions under which its classical solution exists.

Keywords: wave equation, mixed problem, method of characteristics, classical solution, matching conditions, conjugation conditions, discontinuous conditions, curvilinear domain

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КЛАССИЧЕСКОЕ РЕШЕНИЕ СМЕШАННОЙ ЗАДАЧИ В КРИВОЛИНЕЙНОЙ ПОЛУПОЛОСЕ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ С РАЗРЫВНЫМИ НАЧАЛЬНЫМИ УСЛОВИЯМИ

Аннотация. Для одномерного волнового уравнения рассматривается смешанная задача в криволинейной полуполосе. Начальные условия имеют разрыв первого рода в одной точке. Смешанная задача моделирует задачу о продольном ударе по конечному упругому стержню с подвижной границей. Решение строится методом характеристик в явном аналитическом виде. Для рассматриваемой задачи доказывается единственность решения и устанавливаются условия, при которых существует ее классическое решение.

Ключевые слова: волновое уравнение, смешанная задача, метод характеристик, классическое решение, условия согласования, условия сопряжения, разрывные условия, криволинейная область

Для цитирования. Корзюк, В. И. Классическое решение смешанной задачи в криволинейной полуполосе для волнового уравнения с разрывными начальными условиями / В. И. Корзюк, Я. В. Рудько, В. В. Колячко // Доклады Национальной академии наук Беларуси. – 2025. – Т. 69, № 4. – С. 271–278. https://doi.org/10.29235/1561-8323-2025-69-4-271-278

Statement of the problem. In the curvilinear domain $Q = \{(t, x) : t \in (0, \infty) \wedge x \in (\gamma(t), l)\}$, where l is a positive real number, of two independent variables $(t, x) \in \overline{Q} \subset \mathbb{R}^2$, for the wave equation

$$(\partial_t^2 - a^2 \partial_x^2) u(t, x) = f(t, x), \ (t, x) \in Q, \tag{1}$$

we consider the following mixed problem with the initial conditions

$$u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x)+\begin{cases}0, & x\in[0,l),\\ v, & x=l,\end{cases}\ x\in[0,l], \tag{2}$$

and the boundary conditions

$$u(t,\gamma(t))=\mu_1(t),\ \ (\partial_t^2+b\partial_x)u(t,l)=\mu_2(t),\ \ t\in[0,\infty), \tag{3}$$

where a, v, and b are real numbers, $a>0$ for definiteness, f is a function given on the set $\bar{Q}$, φ and ψ are some real-valued functions defined on the segment $[0, l]$, and μ_1 and μ_2 are some real-valued functions defined on the half-line $[0,\infty)$. We also assume that

$$\gamma\in C^1([0,\infty)),\ \ \gamma'(t)\in(-a,a)\ \text{ for all }\ t\in[0,\infty),\ \ \lim_{t\to+\infty}\gamma(t)\pm at=\pm\infty, \tag{4}$$

and that the curves $x=\gamma(t)$ and $x=l$ do not intersect.

The mixed problem (1)–(3) models the following problem from the theory of longitudinal impact. Suppose that an elastic finite homogeneous rod of constant cross-section, where the left moving boundary $x=\gamma(t)$ is fixed, is subjected at the initial moment $t=0$ to an impact at the end $x=l$ by a load that sticks to the rod. Furthermore, we assume that an external volumetric force is applied to the rod, that the displacements of the rod and the rate of their change at the initial moment $t=0$ are not equal to zero, and that no shock waves present within the rod. Then, neglecting both the weight of the rod as a force and its possible vertical displacements, the displacements $u(t,x)$ of the rod satisfy the mixed problem (1)–(3), where $a=\sqrt{E\rho^{-1}}$, $b=SEM^{-1}$, where $E>0$ is Young's modulus of the rod material, $\rho>0$ is the density of the rod material, $S>0$ is the cross-sectional area of the rod, $M>0$ is the mass of the impacting load, $-v$ is the velocity of the impacting load, μ_2 is the external force applying to the end of the rod divided by the mass of the impacting load. The quantity $\mu_1(t)$ has a physical meaning of the external force acting on the end of the rod, $\mu_2(t)$ has a physical meaning of function that defines the movement of the end $x=0$ of the rod in the longitudinal direction, divided by the mass of the impacting load. The function f is the external volumetric force divided by ρ.

In the case $\gamma(t)=0$, $\mu_1=\mu_2\equiv 0$, $\varphi=\psi\equiv 0$, and $f\equiv 0$, a unique generalized solution of the problem (1)–(3) was obtained in the work [1], although the physical correctness of the solution was not proven. A similar mixed problem, featuring a boundary condition $(\partial_t^2+b\partial_x+c)u(t,l)=0$ instead of $(\partial_t^2+b\partial_x)u(t,l)=0$, was studied in the work [2], where once more a unique generalized solution was constructed and its physical correctness was not established. In the case of smooth data, i. e., $v=0$, and a regular half-strip, i. e., $\gamma\equiv 0$, the problem (1)–(3) has been studied using both Fourier series [3] and the method of characteristics [4]. Similar problems in curvilinear domains have been considered in the works [5–7].

Curvilinear half-strip. Let us note some properties of the domain Q in which the problem is considered.

A s s e r t i o n 1. *Let* $(t_0,x_0)\in Q$. *Then the value* x_0+at_0 *is nonnegative under the conditions* (*4*).

A s s e r t i o n 2. *Let* $\alpha\in[0,\infty)$. *Then the equation* $\gamma(t)+at=\alpha$ *has a unique solution under the conditions (4).*

A s s e r t i o n 3. *Let* $\alpha\in(-\infty,0]$. *Then the equation* $\gamma(t)-at=\alpha$ *has a unique solution under the conditions (4).*

A s s e r t i o n 4. *Let* $(t_0,x_0)\in Q$. *Then the curve* $(t,\gamma(t))$ *intersects the line* $x+at=x_0+at_0$ *at a single point under the conditions (4).*

A s s e r t i o n 5. *Let* $(t_0,x_0)\in Q$ *and* $x_0-at_0\le 0$. *Then the curve* $(t,\gamma(t))$ *intersects the line* $x-at=x_0-at_0$ *at a single point under the conditions (4).*

The proofs of Assertions 1–5 are given in the article [7].

Consider the following functions

$$\gamma_+:[0,\infty)\ni t\mapsto\gamma(t)+at,\ \ \gamma_-:[0,\infty)\ni t\mapsto\gamma(t)-at.$$

We also need the inverse of the functions γ_+ and γ_-, which will be denoted by the symbols Φ_+ and Φ_-, respectively, i. e., $\Phi_+(\gamma(t)+at)=t$ and $\Phi_-(\gamma(t)-at)=t$. Such functions are guaranteed to exist by Assertions 2 and 3. According to the inverse function theorem, we derive the formulas

$$\Phi'_-(t)=\frac{1}{\gamma'(\Phi_-(t))-a},\ \Phi''_-(t)=-\frac{\gamma''(\Phi_-(t))}{(\gamma'(\Phi_-(t))-a)^3},\ t\in[0,\infty), \tag{5}$$

$$\Phi'_+(t)=\frac{1}{\gamma'(\Phi_+(t))+a},\ \Phi''_+(t)=-\frac{\gamma''(\Phi_+(t))}{(\gamma'(\Phi_+(t))+a)^3},\ t\in[0,\infty). \tag{6}$$

Note that the representations (5) and (6), along with condition (4), imply that Φ_+ is an increasing function while Φ_- is a decreasing function.

Auxiliary problem. Consider the following simple case

$$v=0. \tag{7}$$

The solution u of the problem (1)–(3), (7) has the form

$$u(t,x)=w(t,x)+g(x-at)+p(x+at), \tag{8}$$

where w is a particular solution of (1). We can take it from the paper [6], it satisfies the homogeneous initial conditions

$$w(0,x)=\partial_t w(0,x)=0,\ x\in[0,l],$$

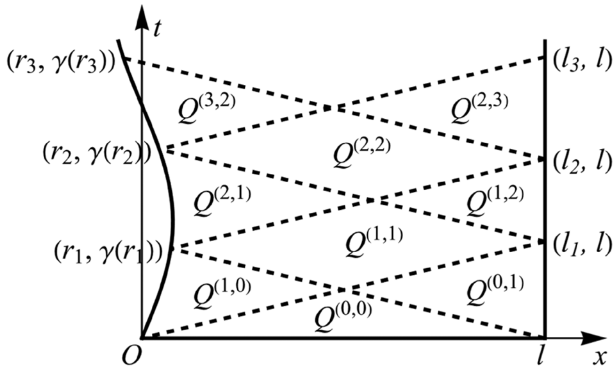
and belongs to the class $C^2(\overline{Q})$ if, for example, $f\in C^1(\overline{Q})$. Moreover, $\partial_t^2 w(0,x)=f(0,x)$ holds for all $x\in[0,l]$.

So, we want to find closed-form expressions for the functions g and p. To do this, we partition the domain $\overline{Q}$ according to the following formulas (for clarity see Figure):

$$\begin{gathered}
Q^{(0,0)}=Q\cap\{(t,x):x-at\in[0,l]\wedge x+at\in[0,l]\},\\
Q^{(1,0)}=Q\cap\{(t,x):x-at\in[\gamma_-(r_1),0]\wedge x+at\in[0,l]\},\\
Q^{(0,1)}=Q\cap\{(t,x):x-at\in[0,l]\wedge x+at\in[l,l+al_1]\},\\
Q^{(i,j)}=Q\cap\{(t,x):x-at\in[\gamma_-(r_i),\gamma_-(r_{i-1})]\wedge x+at\in[l+al_{j-1},l+al_j]\},
\end{gathered} \tag{9}$$

where $r_0=l_0=0;\ l_i=r_{i-1}+a^{-1}(l-\gamma(r_{i-1}));\ r_i=\Phi_+(l+al_{i-1})$.

From the geometric considerations and Assertions 1–5, it is straightforward to demonstrate the correctness of the partitioning (9) of the domain Q.



Partitioning of the domain Q

We will look for the function as a piecewise-defined, i. e.,

$$u(t,x)=u^{(i,j)}(t,x),\ (t,x)\in\overline{Q^{(i,j)}}. \tag{10}$$

Because of (8)–(10), we can write

$$u^{(i,j)}(t,x)=w(t,x)+g^{(i)}(x-at)+p^{(j)}(x+at),\ (t,x)\in\overline{Q^{(i,j)}}. \tag{11}$$

We determine the functions $g^{(0)}$ and $p^{(0)}$ from the Cauchy conditions (2):

$$g^{(0)}(x)=\frac{\varphi(x)}{2}-\frac{1}{2a}\int_0^x\psi(\xi)d\xi+C_1,\ x\in[0,l], \tag{12}$$

$$p^{(0)}(x)=\frac{\varphi(x)}{2}+\frac{1}{2a}\int_0^x\psi(\xi)d\xi-C_1,\ x\in[0,l], \tag{13}$$

where C_1 is a real number. The function $g^{(i)}$ for $i\in\mathbb{N}$ can be defined from the Dirichlet boundary condition (3) on the curve $x=\gamma(t)$. We substitute (11), where $Q^{(i,j)}=Q^{(i,i-1)}$, into (3) and obtain

$$w(t,\gamma(t))+g^{(i)}(\gamma(t)-at)+p^{(i-1)}(\gamma(t)+at)=\mu_1(t),\ t\in[r_{i-1},r_i],\ i\in\mathbb{N}.$$

Changing the variable $z=\gamma(t)-at$, i. e., $t=\Phi_-(z)$, results in the equation

$$w(\Phi_-(z),\gamma(\Phi_-(z)))+g^{(i)}(z)+p^{(i-1)}(\gamma(\Phi_-(z))+a\Phi_-(z))=\mu_1(\Phi_-(z)),\ \Phi_-(z)\in[r_{i-1},r_i],\ i\in\mathbb{N},$$

which we can solve to obtain

$$g^{(i)}(z)=\mu_1(\Phi_-(z))-p^{(i-1)}(\gamma(\Phi_-(z))+a\Phi_-(z))-w(\Phi_-(z),\gamma(\Phi_-(z))),\ \Phi_-(z)\in[r_{i-1},r_i],\ i\in\mathbb{N}. \tag{14}$$

The function $p^{(j)}$ for $i\in\mathbb{N}$ can be defined from the boundary condition (3) on the line $x = l$. Again, we substitute (11), where $Q^{(i,j)}=Q^{(j-1,j)}$, into (3) and get

$$\begin{gathered}a^2D^2g^{(j-1)}(l-at)+a^2D^2p^{(j)}(l+at)+\\+b(Dg^{(j-1)}(l-at)+Dp^{(j)}(l+at)+\partial_x w(t,l))+\partial_t^2 w(t,l)=\mu_2(t),\\ t\in[l_{i-1},l_i],\ j\in\mathbb{N},\end{gathered} \tag{15}$$

where D is the Newton–Leibniz operator. Changing the variable $t=a^{-1}(z-l)$ transforms (15) into

$$\begin{gathered}a^2D^2p^{(j)}(z)+bDp^{(j)}(z)=\mu_2\left(\frac{z-l}{a}\right)-a^2D^2g^{(j-1)}(2l-z)-\\-bDg^{(j-1)}(2l-z)-b\partial_x v\left(\frac{z-l}{a},l\right)-\partial_t^2 v\left(\frac{z-l}{a},l\right),\\ z\in[l+al_{i-1},l+al_i],\ j\in\mathbb{N}.\end{gathered} \tag{16}$$

We solve (16) and obtain

$$\begin{gathered}p^{(j)}(z)=p^{(j)}(l+al_{j-1}+0)+\int_{l+al_{j-1}}^{z}\exp\left(\frac{b(l+al_{j-1}-\eta)}{a^2}\right)\times\\ \times\left(Dp^{(j)}(l+al_{j-1}+0)+\int_{l+al_{j-1}}^{\eta}a^{-2}\exp\left(\frac{b(\xi-l-al_{i-1})}{a^2}\right)\left\{\mu_2\left(\frac{\xi-l}{a}\right)-a^2D^2g^{(j-1)}(2l-\xi)-\right.\right.\\ \left.\left.-bDg^{(j-1)}(2l-\xi)-b\partial_x v\left(\frac{\xi-l}{a},l\right)-\partial_t^2 v\left(\frac{\xi-l}{a},l\right)\right\}d\xi\right)d\eta,\ z\in[l+al_{j-1},l+al_j],\ j\in\mathbb{N}.\end{gathered} \tag{17}$$

We choose the values $p^{(j)}(l+al_{j-1}+0)$ and $Dp^{(j)}(l+al_{j-1}+0)$ in the representation (17) by continuity, i. e.,

$$p^{(j)}(l+al_{j-1}+0)=p^{(j-1)}(l+al_{j-1}-0),\ Dp^{(j)}(l+al_{j-1}+0)=Dp^{(j-1)}(l+al_{j-1}-0),\ j\in\mathbb{N}. \quad (18)$$

According to formulas (14), (17), and (18), the following relations hold for all $i\in\mathbb{N}$:

$$\Delta_g^i = g^{(i+1)}(\gamma_-(r_i)) - g^{(i)}(\gamma_-(r_i)) = p^{(i-1)}(\gamma_+(r_i)) - p^{(i)}(\gamma_+(r_i)) = 0,$$

$$\tilde{\Delta}_g^i = Dg^{(i+1)}(\gamma_-(r_i)) - Dg^{(i)}(\gamma_-(r_i)) = -\frac{a+\gamma'(r_i)}{a-\gamma'(r_i)}(Dp^{(i-1)}(\gamma_+(r_i)) - Dp^{(i)}(\gamma_+(r_i))) = 0,$$

$$\approx{\Delta}_g^i = D^2g^{(i+1)}(\gamma_-(r_i)) - D^2g^{(i)}(\gamma_-(r_i)) = \frac{(a+\gamma'(r_i))^2}{(a-\gamma'(r_i))^2}(D^2p^{(i-1)}(\gamma_+(r_i)) - D^2p^{(i)}(\gamma_+(r_i))),$$

$$\Delta_p^i = p^{(i+1)}(l+al_i) - p^{(i)}(l+al_i) = 0,\ \tilde{\Delta}_p^i = Dp^{(i+1)}(l+al_i) - Dp^{(i)}(l+al_i) = 0,$$

$$\approx{\Delta}_p^i = D^2p^{(i+1)}(l+al_i) - D^2p^{(i)}(l+al_i) = -D^2g^{(i)}(l-al_i) + D^2g^{(i-1)}(l-al_i) - -\frac{b}{a^2}Dg^{(i)}(l-al_i) + \frac{b}{a^2}Dg^{(i-1)}(l-al_i).$$

By virtue of the expressions $l_i = r_{i-1} + a^{-1}(l-\gamma(r_{i-1}))$ and $r_i = \Phi_+(l+al_{i-1})$ we have

$$\Delta_g^i = \tilde{\Delta}_g^i = \Delta_p^i = \tilde{\Delta}_p^{i-1} = 0,\ \approx{\Delta}_g^i = -\frac{(a+\gamma'(r_i))^2}{(a-\gamma'(r_i))^2}\approx{\Delta}_p^{i-1},\ \approx{\Delta}_p^i = -\approx{\Delta}_g^{i-1} - \frac{b}{a^2}\tilde{\Delta}_g^{i-1},\ i\in\mathbb{N}. \quad (19)$$

The base of the recurrence relations (19) can be computed using the representations (12), (13), (15), (17), and (18). So, after some simple calculations, we get

$$\Delta_g^0 = \delta_0 = \mu_1(0) - \varphi(0),\ \tilde{\Delta}_g^0 = \delta_1 = \frac{\psi(0) + \gamma'(0)\varphi'(0) - \mu_1'(0)}{a-\gamma'(0)},$$

$$\approx{\Delta}_g^i = \delta_2 = \frac{1}{(a-\gamma'(0))^3}((\mu_1'(0) - \psi(0))\gamma''(0) - a^3\varphi''(0) + a^2\gamma'(0)\varphi''(0) + \gamma'(0)\times$$
$$\times(f(0,0) + 2\gamma'(0)\psi'(0) + \gamma'(0)^2\varphi''(0) - \mu_1''(0)) -$$
$$-a(f(0,0) + 2\gamma'(0)\psi'(0) + \varphi'(0)\gamma''(0) + \gamma'(0)^2\varphi''(0) - \mu_1''(0)), \quad (20)$$

$$\Delta_p^0 = \rho_0 = 0,\ \tilde{\Delta}_p^0 = \rho_1 = 0,\ \approx{\Delta}_p^0 = \rho_2 = -\frac{f(0,l) - \mu_2(0) + b\varphi'(0) + a^2\varphi''(l)}{a^2}.$$

The following assertion holds.

A s s e r t i o n 6. *Let the smoothness conditions*

$$\varphi\in C^2([0,l]),\ \psi\in C^1([0,l]),\ \mu_1\in C^2([0,\infty)),\ \mu_2\in C([0,\infty)),\ \gamma\in C^2([0,\infty)), \quad (21)$$

be satisfied. Then the functions g and p, defined by the formulas (12)–(14), (17), (18), and

$$g(z) = g^{(0)}(z),\ z\in[0,l],\ g(z) = g^{(i)}(z),\ \Phi_-(z)\in[r_{i-1}, r_i],\ i\in\mathbb{N};$$
$$p(z) = p^{(0)}(z),\ z\in[0,l],\ p(z) = p^{(i)}(z),\ z\in[l+al_{j-1}, l+al_j],\ j\in\mathbb{N}, \quad (22)$$

are twice continuously differentiable if and only if the following matching conditions are satisfied

$$\mu_1(0) - \varphi(0) = 0, \quad (23)$$

$$\mu_1'(0)-\psi(0)+\gamma'(0)\varphi'(0)=0, \tag{24}$$

$$\mu_1''(0)-(a^2+(\gamma'(0))^2)\varphi''(0)-f(0,0)-2\gamma'(0)\psi'(0)-\gamma''(0)\varphi'(0)=0, \tag{25}$$

$$\mu_2(0)-f(0,l)-b\varphi'(0)-a^2\varphi''(l)=0. \tag{26}$$

The proof follows from the formulas (12)–(14), (17)–(20).

A s s e r t i o n 7. *The functions g and p defined by the formulas (12)–(14), (17), (18), and (22) are of the form* $g=\tilde{g}+C_1$, $p=\tilde{p}-C_1$, *where* $\tilde{g}$ *and* $\tilde{p}$ *are some functions not depending on the constant* C_1.

The proof can be easily done by the method of mathematical induction (see, e. g., [8, p. 179]).

T h e o r e m 1. *Let the smoothness conditions (21) and*

$$f\in C^1(\overline{Q}) \tag{27}$$

be satisfied. The mixed problem (1)–(3), (7) has a unique solution in the class $C^2(\overline{Q})$ if and only if the matching conditions (23)–(26) are satisfied. This solution is determined by the formulas (10)–(14), (17), (18).

P r o o f. The existence of the solution follows from Assertions 6 and 7. The uniqueness of the solution follows from the construction and Assertion 7, as it has been derived from the general solution.

Main problem. Since in the general case $\psi\notin C^1([0,l])$, the problem (1)–(3) has no solution belonging to the class $C^2(\overline{Q})$, i. e., the problem (1)–(3) does not have a global classical solution defined on the set $\overline{Q}$. However, it is possible to define a classical solution on a smaller set $\overline{Q}\setminus\Gamma$ that will satisfy Eq. (1) on the set $\overline{Q}\setminus\Gamma$ in the standard sense and some additional conjugation conditions on the set Γ.

D e f i n i t i o n. *A function u is a classical solution of the problem (1)–(3) if it is representable in the form* $u=u_1+u_2$, *where* u_1 *is a classical solution of the problem (1)–(3) with* $\nu=0$ *and* u_2 *satisfies Eq. (1) with* $f\equiv 0$, *the initial conditions* $u_2(0,x)=\partial_t u_2(0,x)=0$, $x\in[0,l]$, *the boundary conditions (3) with* $\mu_1=\mu_2\equiv 0$, *and the following matching conditions*

$$[(u_2)^+-(u_2)^-](t,x=\gamma_-(r_i)+at)=0, \tag{28}$$

$$[(u_2)^+-(u_2)^-](t,x=l+al_i-at)=0,\ i\in\{0\}\cup\mathbb{N}, \tag{29}$$

$$[(\partial_t u_2)^+-(\partial_t u_2)^-](t,x=l+al_i-at)=\begin{cases}\nu, & i\equiv 0\ (\mathrm{mod}\ 2),\\ 0, & i\equiv 1\ (\mathrm{mod}\ 2),\end{cases}\ i\in\{0\}\cup\mathbb{N}. \tag{30}$$

T h e o r e m 2. *Let the smoothness conditions (21) and (27) be satisfied. The mixed problem (1)–(3) has a unique solution in the sense of Definition if and only if the matching conditions (23)–(26) are satisfied.*

P r o o f. According to Theorem 1, under the smoothness conditions (21) and (27), the "smooth" part of the solution, i. e., the function u_1 from Definition, exists and is unique if and only if the conditions (23)–(26) are satisfied. The "discontinuous" part of the solution, i. e., the function u_2 from Definition, can be defined by the formula

$$u_2(t,x)=g_*^{(i)}(x-at)+p_*^{(j)}(x+at),\ (t,x)\in\overline{Q^{(i,j)}}, \tag{31}$$

where

$$g_*^{(0)}(x)=p_*^{(0)}(x)=0,\ x\in[0,l], \tag{32}$$

$$g_*^{(i)}(z)=-p_*^{(i-1)}(\gamma(\Phi_-(z))+a\Phi_-(z)),\ \Phi_-(z)\in[r_{i-1},r_i],\ i\in\mathbb{N}, \tag{33}$$

$$p_*^{(j)}(z) = p_*^{(j-1)}(l + al_{j-1} - 0) + \int_{l+al_{j-1}}^{z} \exp\left(\frac{b(l + al_{j-1} - \eta)}{a^2}\right) \times$$
$$\times \left\{ Dp_*^{(j-1)}(l + al_{j-1} - 0) + \begin{cases} a^{-1}v, & j \equiv 1 \ (\mathrm{mod}\ 2), \\ 0, & j \equiv 0 \ (\mathrm{mod}\ 2), \end{cases} - \right. \tag{34}$$
$$\left. - \int_{l+al_{j-1}}^{\eta} a^{-2} \exp\left(\frac{b(\xi - l - al_{i-1})}{a^2}\right)(a^2D^2g_*^{(j-1)}(2l - \xi) + bDg_*^{(j-1)}(2l - \xi))d\xi \right\} d\eta,$$
$$z \in [l + al_{j-1}, l + al_j], \ j \in \mathbb{N}.$$

The formulas (31)–(34) can be derived analogously to (10)–(14), (17), (18). As in Theorem 1, the uniqueness of the solution is followed by construction since we constructed it from the general solution.

Now, let us provide justification for the choice of the conjugation conditions (28)–(30) on the basis of physical considerations. We derive the conditions (28) and (29) from the continuity. Therefore, we only need to show the correctness of the condition (30). At the initial moment $t = 0$, the rod is subjected to an impact at the end $x = l$. It generates a shock wave that spreads along the characteristic $x + at = l$. Its velocity must satisfy the following condition

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l - at) = v.$$

For the derivation of the previous equality, we refer the reader to our paper [9]. Furthermore, at the moment when the end point of the rod is reached, the wave is immediately reflected and propagates along the characteristic with a speed that we do not set but which we can calculate as follows

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = \gamma_-(r_1) - at) = v\frac{a + \gamma'(r_1)}{a - \gamma'(r_1)}.$$

The interaction with the moving end of the rod changes its velocity. Furthermore, according to (30), after reflecting from the end $x = l$ of the rod, this wave will travel with the velocity

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l + al_2 - at) = v.$$

However, since the wave propagates at the same speed in elastic rods, the condition $\gamma'(r_1) = 0$ must be satisfied for the solution of the problem (1)–(3) to be correct in the sense of Definition. Following this scheme, we prove the physical correctness of the condition

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l + al_i - at) = v, \ i \equiv 0 \ (\mathrm{mod}\ 2), \ i \in \{0\} \cup \mathbb{N},$$

if $\gamma'(r_j) = 0$ for all $j \equiv 1 \ (\mathrm{mod}\ 2)$. Since the end $x = 0$ of the rod was not hit, shock waves should not propagate along the characteristics $x = l + al_i - at$ and $x = \gamma_-(r_{i-1}) - at$, where $i \equiv 1 \ (\mathrm{mod}\ 2)$, $i \in \{0\} \cup \mathbb{N}$. It implies

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = l + al_i - at) = 0, \ i \equiv 1 \ (\mathrm{mod}\ 2), \ i \in \{0\} \cup \mathbb{N},$$

and

$$[(\partial_t u)^+ - (\partial_t u)^-](t, x = \gamma_-(r_i) - at) = 0, \ i \equiv 0 \ (\mathrm{mod}\ 2), \ i \in \{0\} \cup \mathbb{N}.$$

The latter can be verified by the formulas (31)–(34) and the fact that $u_1 \in C^2(\overline{Q})$. It brings us to the following statement.

A s s e r t i o n 8. *Let the conditions (21), (27), and (23)–(26) be satisfied. Then a solution of the problem (1)–(3) in the sense of Definition is physically correct if the following condition*

$$\gamma'(r_j) = 0, \ j \in \{0\} \cup \mathbb{N}, \ j \equiv 1 \ (\mathrm{mod}\ 2),$$

is satisfied.

Conclusions. In the present paper, we have obtained the necessary and sufficient conditions under which a unique classical solution of a mixed problem exists for the wave equation with discontinuous conditions in a curvilinear half-strip. We have constructed the solution in an implicit analytical form. We have proposed a method for constructing solutions to mixed problems for hyperbolic equations with discontinuous conditions in curvilinear domains.

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