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MONOTONE DIFFERENCE SCHEMES FOR QUASILINEAR PARABOLIC EQUATIONS WITH REACTION COEFFICIENTS OF ARBITRARY SIGN

Abstract. This paper establishes continuous and discrete two-sided estimates for a class of quasilinear parabolic convection–diffusion–reaction equations with unbounded nonlinearities and reaction coefficients of arbitrary sign. For nonpositive reaction coefficients, rigorous continuous two-sided bounds are derived for classical solutions, together with an a priori estimate in the strong C-norm. These results extend, in a structurally coherent manner, the corresponding theory available for linear parabolic problems. For reaction coefficients of arbitrary sign, second-order unconditionally monotone schemes are constructed. By applying the Ladyzhenskaya transformation $u = ve^{\lambda t}$, alternative discrete two-sided estimates are derived for the numerical solution and shown to be fully consistent with their continuous counterparts. Finally, convergence of the developed schemes is rigorously proved in the L_2-norm.

Keywords: maximum principle, two-sided estimates, quasilinear parabolic convection–diffusion–reaction equation, reaction coefficient of arbitrary sign, convergence, monotone difference scheme

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МОНОТОННЫЕ РАЗНОСТНЫЕ СХЕМЫ ДЛЯ КВАЗИЛИНЕЙНЫХ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ С КОЭФФИЦИЕНТАМИ РЕАКЦИИ ПРОИЗВОЛЬНОГО ЗНАКА

Аннотация. Установлены непрерывные и дискретные двусторонние оценки для класса квазилинейных параболических уравнений конвекции–диффузии–реакции с неограниченными нелинейностями и коэффициентами реакции произвольного знака. Для неположительных коэффициентов реакции выведены строгие непрерывные двусторонние оценки для классических решений, а также априорная оценка в сильной C-норме. Полученные результаты структурно непротиворечивым образом расширяют соответствующую теорию, доступную для линейных параболических задач. Для коэффициентов реакции произвольного знака построены безусловно монотонные схемы второго порядка. С помощью преобразования Ладыженской $u = ve^{\lambda t}$ выведены альтернативные дискретные двусторонние оценки для численного решения, которые, как показано, полностью согласуются с их непрерывными аналогами. Сходимость разработанных схем строго доказана в L_2-норме.

Ключевые слова: принцип максимума, двусторонние оценки, квазилинейное параболическое уравнение конвекции–диффузии–реакции, коэффициент реакции произвольного знака, сходимость, монотонная разностная схема

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Introduction. Partial differential equations provide a fundamental mathematical framework for modeling diffusion, transport, and reaction processes in natural and engineering sciences. Among them, parabolic equations are particularly important due to their role in describing heat conduction, subsurface flow, and nonlinear diffusion phenomena. A central qualitative feature of such problems is the validity of continuous two-sided estimates, which ensure that the solution remains bounded by functions determined by the initial and boundary data as well as the source term. This property reflects the intrinsic smoothing character of diffusion and guarantees stability and physical admissibility.

For linear parabolic equations, continuous two-sided estimates have been rigorously established and extensively applied in the study of existence, uniqueness, and stability; see [1–3]. In particular, [2] provides a precise formulation that has proved useful in both theoretical and numerical contexts. However, when nonlinear diffusion or convection terms are present, especially with solution-dependent coefficients, classical comparison techniques are no longer directly applicable, and the derivation of such estimates becomes substantially more involved.

At the discrete level, preserving qualitative properties of the differential model is essential for reliable simulations. Discrete two-sided estimates prevent artificial oscillations and spurious extrema. Early contributions can be found in [4; 5], with further developments in [2; 6; 7]. Related studies on monotone schemes and the maximum principle include [8–15].

Nevertheless, significantly fewer results address reaction coefficients of arbitrary sign. Motivated by this gap, the present work constructs unconditionally monotone finite-difference schemes (FDSs) for quasilinear parabolic equations with unbounded nonlinearity and sign-indefinite reaction terms, employing the exponential transformation technique from [1], and establishes consistent continuous and discrete two-sided estimates together with convergence results.

Problem statement. In the domain $\bar{Q}_T = \{(x,t): x \in \bar{\Omega},\, t \in [0,T]\}$, $\bar{\Omega} = \{x : 0 \le x \le l\} = \Omega \cup \{0, l\}$ we study the initial-boundary value problem for the quasilinear parabolic convection–diffusion–reaction equation with unbounded nonlinearity and reaction coefficient of arbitrary sign

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + v(u)\frac{\partial u}{\partial x} + q(x,t)u + f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

where $|v(u)| \le c_0$, with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega},$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t > 0. \tag{2}$$

In this paper, we assume that problem (1), (2) admits a unique classical solution. Furthermore, the functions $q(x, t)$ and $f(x, t)$ are assumed to be continuous in $\bar{Q}_T$. A characteristic feature of problems with unbounded nonlinearity is that the fundamental properties (positivity, boundedness, etc.) imposed on the nonlinear coefficient $k(u)$ is required not for all values of the parameter u, but only within the range of the exact solution $\bar{D}_u$ or in a small neighborhood thereof [15]. In what follows, we assume that the parabolicity condition for Eq. (1) is satisfied on the solution [2], namely,

$$0 < k_1 \le k(u) \le k_2, \quad \forall u \in \bar{D}_u, \tag{3}$$

where k_1 and k_2 are positive constants. Since the coefficient $k(u)$ is smooth, it follows that there exits a constant L such that $|k'(u)| \le L$, $\forall u \in \bar{D}_u$.

Case of nonpositive reaction coefficients. Consider the case of $q(x, t) \le 0$. The following continuous two-sided estimates for the classical solution were rigorously established, together with an a priori estimate in the strong C-norm. These estimates are structurally analogous to the corresponding results for the linear problem [3], thereby extending the known theory to the quasilinear setting.

T h e o r e m 1. *Let condition $q(x, t) \le 0$ and condition (3) be satisfied. Then the classical solution $u(x, t)$ of problem (1), (2) satisfies the following two-sided estimates*

$$m_1 \le u(x,t) \le m_2, \quad (x,t) \in \bar{Q}_T,$$

where m_1, m_2 are constants defined by

$$m_1 = \min\left\{0, \min_{(x,t)\in\partial Q_T}\{u_0(x), \mu_1(t), \mu_2(t)\}\right\} + T\min\left\{0, \min_{(x,t)\in\bar{Q}_T} f(x,t)\right\},$$

$$m_2 = \max\left\{0, \max_{(x,t)\in\partial Q_T}\{u_0(x), \mu_1(t), \mu_2(t)\}\right\} + T\max\left\{0, \max_{(x,t)\in\bar{Q}_T} f(x,t)\right\}.$$

C o r o l l a r y. *Let u be a classical solution of the problem (1), (2). Then the following estimate in C-norm holds*

$$\|u\|_{C(\bar{Q}_T)} \le \max\left\{\|u_0\|_{C(\bar{\Omega})}, \|\mu_1\|_{C(0,T)}, \|\mu_2\|_{C(0,T)}\right\} + T\|f\|_{C(\bar{Q}_T)},$$

where

$$\|g\|_{C(\bar{Q}_T)} = \max_{(x,t)\in\bar{Q}_T}|g(x,t)|, \quad \|g\|_{C(\bar{\Omega})} = \max_{x\in\bar{\Omega}}|g(x)|, \quad \|g\|_{C(0,T)} = \max_{t\in[0,T]}|g(t)|.$$

Case of reaction coefficients of arbitrary sign. Let $t_1 \le T$ and $Q_{t_1} = \{(x,t) \in Q_T : t \le t_1\}$. Using the technique from [1], we proved two-sided estimates for the exact solution of problem (1), (2).

T h e o r e m 2. *Let condition (3) be satisfied. Then for solution $u(x, t)$ of problem (1), (2) at any point $(x, t) \in \bar{Q}_T$ the following two-sided estimates are valid*

$$m_3 \le u(x, t_1) \le m_4, \tag{4}$$

where m_3, m_4 are constants defined by

$$m_3 = \sup_{\lambda>\lambda_0} e^{\lambda t_1}\min\left\{0, \min_{[0,l]} u_0(x), \min_{[0,t_1]}(\mu_1(t)e^{-\lambda t}), \min_{[0,t_1]}(\mu_2(t)e^{-\lambda t}), \frac{1}{\lambda-\lambda_0}\min_{Q_{t_1}}(f(x,t)e^{-\lambda t})\right\},$$

$$m_4 = \inf_{\lambda>\lambda_0} e^{\lambda t_1}\max\left\{0, \max_{[0,l]} u_0(x), \max_{[0,t_1]}(\mu_1(t)e^{-\lambda t}), \max_{[0,t_1]}(\mu_2(t)e^{-\lambda t}), \frac{1}{\lambda-\lambda_0}\max_{Q_{t_1}}(f(x,t)e^{-\lambda t})\right\},$$

with $\lambda_0 = \max_{Q_T} q(x,t)$.

Nonstandard maximum principle for difference schemes in the canonical form. Let in the n-dimensional Euclidian space a finite number of points of the grid Ω_h is given. To each point $x \in \Omega_h$ we associate one and only one stencil $\mathcal{M}(x)$ – a subset of Ω_h, containing this point. The set $\mathcal{M}'(x) = \mathcal{M}(x) \setminus x$ is called *neighborhood* of the point x. Let the functions $A(x)$, $B(x, \xi)$, $F(x)$ be given at $x \in \Omega_h$, $\xi \in \Omega_h$ and they take real values. Next, to each point $x \in \Omega_h$ corresponds one and only one equation of the form [8]

$$A(x)y(x) = \sum_{\xi\in\mathcal{M}'(x)} B(x,\xi)y(\xi) + F(x), \quad x \in \Omega_h, \tag{5}$$

which is called the *canonical form* of the FDS. According to [8], the point x is called a *boundary grid node*, if Dirichlet condition is posed

$$y(x) = \mu(x), \quad x \in \gamma, \tag{6}$$

where γ is the set of the boundary nodes, $\bar{\Omega}_h = \Omega_h \cup \gamma$. We will assume the fulfilment of the usual positivity conditions for the coefficients

$$A(x) > 0, \quad B(x,\xi) > 0 \quad \text{for all} \quad \xi \in \mathcal{M}'(x), \tag{7}$$

$$D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi) > 0. \tag{8}$$

L e m m a [12]. *Assume that the positivity conditions (7), (8) are satisfied. Then the maximum and minimum values of the solution of the FDS (5), (6) belong to the range of the input data*

$$\min\left\{\min_{x\in\gamma} \mu(x),\ \min_{x\in\Omega_h} \frac{F(x)}{D(x)}\right\} \le y(x) \le \max\left\{\max_{x\in\gamma} \mu(x),\ \max_{x\in\Omega_h} \frac{F(x)}{D(x)}\right\},\quad x \in \bar{\Omega}_h. \tag{9}$$

Difference schemes and their properties. In the domain $\bar{Q}_T$ we define the usual uniform grid in space and time

$$\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau,\quad \bar{\omega}_h = \{x_i = ih,\ i = \overline{0, N},\ hN = l\},\quad \bar{\omega}_h = \omega_h \cup \{x_0 = 0,\ x_N = l\},$$

$$\bar{\omega}_\tau = \{t_n = n\tau,\ n = \overline{0, N_0},\ \tau N_0 = T\},\quad \bar{\omega}_\tau = \omega_\tau \cup \{t_0 = 0\}.$$

Applying the Samarskii regularization principle [8], for the numerical solution $y(x, t)$ of the problem (1), (2) we construct the FDS of second-order approximation with explicit approximation of the source terms

$$y_t = \varkappa(y)(a(y)\hat{y}_{\bar{x}})_x + b^+(y)a^{(+1)}(y)\hat{y}_x + b^-(y)a(y)\hat{y}_{\bar{x}} + dy + \varphi, \tag{10}$$

$$y(x,0) = u_0(x),\quad x \in \bar{\omega}_h,\quad y_0^{n+1} = \mu_1(t_{n+1}),\quad y_N^{n+1} = \mu_2(t_{n+1}), \tag{11}$$

where

$$\varkappa(y) = \frac{1}{1 + R(y)},\quad R(y) = 0.5h\frac{|v(y)|}{k(y)},\quad b^+(y) = \frac{v^+(y)}{k(y)},\quad b^-(y) = \frac{v^-(y)}{k(y)},$$

$$v^+(y) = 0.5(v(y) + |v(y)|) \ge 0,\quad v^-(y) = 0.5(v(y) - |v(y)|) \le 0,\quad d = q(x_i, t_n),\quad \varphi = f(x_i, t_n).$$

The functionals

$$a_i(y) = 0.5(k(y_{i-1}) + k(y_i)),\quad a^{(+1)}(y) = a_{i+1}(y)$$

are as usually chosen from the second-order approximation condition for the elliptic operator [8]

$$(a(u)\hat{u}_{\bar{x}})_x - \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) = O(h^2 + \tau).$$

Here and further we use the usual notations of the theory of FDSs [8; 9]

$$y = y_i^n = y(x_i, t_n),\quad y_t = (\hat{y} - y)/\tau,\quad \hat{y} = y_i^{n+1},$$

$$g_{\bar{x}} = (g_i - g_{i-1})/h,\quad g_x = (g_{i+1} - g_i)/h,\quad g_\pm = g_{i\pm1} = g(x_{i\pm1}).$$

We show that the solution of the FDS (10), (11) remains within the value range of the exact solution, i. e., $y_i^{n+1} \in \bar{D}_u = [m_3, m_4]$ for all $i = \overline{0, N}$ and $n = \overline{0, N_0 - 1}$.

We introduce an auxiliary grid function $z_i^n = y_i^n e^{-\lambda t_n}$, where λ is an arbitrary constant at this stage. The function $z(x, t)$ satisfies the difference equation in the canonical form (5)

$$\mathcal{C}_i^n z_i^{n+1} = \mathcal{A}_i^n z_{i-1}^{n+1} + \mathcal{B}_i^n z_{i+1}^{n+1} + \mathcal{K}_i^n z_i^n + \mathcal{F}_i^n,\quad i = \overline{1, N-1},$$

$$z_i^0 = u_{0i},\quad z_0^{n+1} = e^{-\lambda t_{n+1}}\mu_1(t_{n+1}),\quad z_N^{n+1} = e^{-\lambda t_{n+1}}\mu_2(t_{n+1}),$$

where

$$\mathcal{A}_i^n = e^{\lambda\tau}\frac{\tau}{h^2}a_i^n(\varkappa_i^n - hb_i^-),\quad \mathcal{B}_i^n = e^{\lambda\tau}\frac{\tau}{h^2}a_{i+1}^n(\varkappa_i^n + hb_i^+),$$

$$\mathcal{C}_i^n = e^{\lambda\tau} + \mathcal{A}_i^n + \mathcal{B}_i^n, \quad \mathcal{K}_i^n = 1 + \tau d_i^n, \quad \mathcal{F}_i^n = \tau f_i^n e^{-\lambda t_n}.$$

The coefficients $\mathcal{D}_i^n$ are defined as follows:

$$\mathcal{D}_i^n = \mathcal{C}_i^n - \mathcal{A}_i^n - \mathcal{B}_i^n - \mathcal{K}_i^n = e^{\lambda\tau} - 1 - \tau d_i^n.$$

Note that the coefficient $\mathcal{K}_i^n \ge 0$ for all $\tau > 0$ if $q_1 := \min\limits_{(x,t)\in\bar{Q}_T} q(x,t) \ge 0$, or for $0 < \tau \le -1/q_1$ if $q_1 < 0$, while the coefficient $\mathcal{D}_i^n > 0$ for $\lambda > \bar{\lambda}_0$, with

$$\bar{\lambda}_0 = \frac{\ln(1+\tau\lambda_0)}{\tau}, \quad \bar{\lambda}_0 \le \lambda_0.$$

Let $\bar{\omega}_{t_{n+1}} = \{(x,t) \in \bar{\omega} : t \le t_{n+1}\}$. The following three cases are possible for the maximum value of the function $z(x, t)$ in the domain $\bar{\omega}_{t_{n+1}}$:

1a) $\max_{\bar{\omega}_{t_{n+1}}} z(x,t)$ is nonpositive (i. e., $z(x, t) \in 0$, $(x,t) \in \bar{\omega}_{t_{n+1}}$);

2a) $\max_{\bar{\omega}_{t_{n+1}}} z(x,t)$ is located on the base $t = 0$ or on the boundary, i. e.,

$$z(x,t) \le \max\left\{\max_{[0,l]} u_0(x),\ \max_{[0,t_{n+1}]}(\mu_1(t)e^{-\lambda t}),\ \max_{[0,t_{n+1}]}(\mu_2(t)e^{-\lambda t})\right\}, \quad (x,t) \in \bar{\omega}_{t_{n+1}};$$

3a) A positive maximum is attained at some interior point $(x^*, t^*) \in \omega_{t_{n+1}}$.

Similarly, there are three possible cases for the minimum value of the function $z(x, t)$ in the domain $\bar{\omega}_{t_{n+1}}$, as follows:

1b) $\min_{\bar{\omega}_{t_{n+1}}} z(x,t)$ is nonnegative (i. e., $z(x, t) \ge 0$, $(x,t) \in \bar{\omega}_{t_{n+1}}$);

2b) $\min_{\bar{\omega}_{t_{n+1}}} z(x,t)$ is located on the base $t = 0$ or on the boundary, i. e.,

$$z(x,t) \ge \min\left\{\min_{[0,l]} u_0(x),\ \min_{[0,t_{n+1}]}(\mu_1(t)e^{-\lambda t}),\ \min_{[0,t_{n+1}]}(\mu_2(t)e^{-\lambda t})\right\}, \quad (x,t) \in \bar{\omega}_{t_{n+1}};$$

3b) A negative minimum is attained at some interior point $(x_*, t_*) \in \omega_{t_{n+1}}$.

If the extremum of the function $z(x, t)$ occurs in the first two cases, namely cases 1a)–2a) and 1b)–2b), then it is straightforward to deduce that $y_i^{n+1} \in \bar{D}_u$, since

$$\min\left\{0, \min_{[0,l]} u_0(x),\ \min_{[0,t_{n+1}]}(\mu_1(t)e^{-\lambda t}),\ \min_{[0,t_{n+1}]}(\mu_2(t)e^{-\lambda t})\right\} \le z(x,t) \le$$

$$\le \max\left\{0, \max_{[0,l]} u_0(x),\ \max_{[0,t_{n+1}]}(\mu_1(t)e^{-\lambda t}),\ \max_{[0,t_{n+1}]}(\mu_2(t)e^{-\lambda t})\right\}, \quad (x,t) \in \bar{\omega}_{t_{n+1}},$$

which is equivalent to

$$m_3 \le \sup_{\lambda > \bar{\lambda}_0} e^{\lambda \bar{t}} \min\left\{0, \min_{[0,l]} u_0(x),\ \min_{[0,t_{n+1}]}(\mu_1(t)e^{-\lambda t}),\ \min_{[0,t_{n+1}]}(\mu_2(t)e^{-\lambda t})\right\} \le y(x,\bar{t}) \le$$

$$\le \inf_{\lambda > \bar{\lambda}_0} e^{\lambda \bar{t}} \max\left\{0, \max_{[0,l]} u_0(x),\ \max_{[0,t_{n+1}]}(\mu_1(t)e^{-\lambda t}),\ \max_{[0,t_{n+1}]}(\mu_2(t)e^{-\lambda t})\right\} \le m_4, \quad \bar{t} \le t_{n+1}.$$

Next, by mathematical induction, we prove that if the extremum of the function $z(x, t)$ occurs in the third case, namely cases 3a) and 3b), then we also obtain $y_i^{n+1} \in \bar{D}_u$.

Obviously, for $n = 0$ we have $y_i^0 = u_{0,i} \in \bar{D}_u$ for all $i = \overline{0, N}$. It follows that $\mathcal{A}_i^0 > 0$, $\mathcal{B}_i^0 > 0$, $\mathcal{D}_i^0 > 0$. Using Lemma on the base of the estimate (9) we get

$$z_i^1 \le \max_{1\le i\le N-1} \frac{\tau f_i^0 e^{-\lambda t_0}}{e^{\lambda\tau} - 1 - \tau q_i^0} \le \max_{\omega_{t_{n+1}}} \frac{\tau f(x,t) e^{-\lambda t}}{e^{\lambda\tau} - 1 - \tau q(x,t)} \le \frac{\tau}{e^{\lambda\tau} - 1 - \tau\lambda_0} \max_{\omega_{t_{n+1}}}(f(x,t)e^{-\lambda t}),$$

and

$$z_i^1 \ge \min_{1\le i\le N-1} \frac{\tau f_i^0 e^{-\lambda t_0}}{e^{\lambda\tau}-1-\tau q_i^0} \ge \min_{\omega_{t_{n+1}}} \frac{\tau f(x,t)e^{-\lambda t}}{e^{\lambda\tau}-1-\tau q(x,t)} \ge \frac{\tau}{e^{\lambda\tau}-1-\tau\lambda_0} \min_{\omega_{t_{n+1}}}(f(x,t)e^{-\lambda t}),$$

which lead to

$$\sup_{\lambda>\bar{\lambda}_0}\left\{\frac{\tau e^{\lambda t_1}}{e^{\lambda\tau}-1-\tau\lambda_0}\min_{\omega_{t_{n+1}}}(f(x,t)e^{-\lambda t})\right\} \le y_i^1 \le \inf_{\lambda>\bar{\lambda}_0}\left\{\frac{\tau e^{\lambda t_1}}{e^{\lambda\tau}-1-\tau\lambda_0}\max_{\omega_{t_{n+1}}}(f(x,t)e^{-\lambda t})\right\}.$$

Since

$$\frac{\tau}{e^{\lambda\tau}-1-\tau\lambda_0} \le \frac{1}{\lambda-\lambda_0} \quad \text{for all} \quad \lambda>\lambda_0\ge\bar{\lambda}_0,\ \ \tau>0,$$

it follows that $y_i^1 \in \bar{D}_u$, i. e., the assertion $y_i^{n+1} \in \bar{D}_u$ holds for $n = 0$.

Assume that, for any arbitrary k ($k \le n$), the inclusion $y_i^k \in \bar{D}_u$ is also true. We need prove that $y_i^{k+1} \in \bar{D}_u$ is true. Indeed, from this assumption we have $\mathcal{A}_i^k > 0$, $\mathcal{B}_i^k > 0$, $\mathcal{C}_i^k > 0$, and for all $t \le t_k$, $x = x_i$ ($i = \overline{0, N}$) the following two-sided estimates hold

$$\frac{\tau}{e^{\lambda\tau}-1-\tau\lambda_0}\min_{\omega_{t_k}}(f(x,t)e^{-\lambda t}) \le z(x,t) \le \frac{\tau}{e^{\lambda\tau}-1-\tau\lambda_0}\max_{\omega_{t_k}}(f(x,t)e^{-\lambda t}). \tag{12}$$

If the extremum of $z(x, t)$ does not attains in the time level $t = t_{k+1}$, then $z(x, t_{k+1})$ satisfies the inequality (12), which yields $y_i^{k+1} \in \bar{D}_u$. On the other hand, if the extremum of $z(x, t)$ attains in the time level $t_*, t^* = t_{k+1}$, then again according to Lemma with the help of the estimate (9) for all $i = \overline{0, N}$, we obtain

$$z_i^{\grave{u}+\grave{u}} \le z(x\ \ ,t^{\ \ +}) \le \frac{\tau f(x^*, t^{\grave{u}})e^{-\lambda\ \ k}}{e^{\lambda\tau}-1-\tau q(x^*, t_k)} \le \max_{\omega_{t_{k+1}}}\frac{\tau f(x,t)e^{-\lambda}}{e^{\lambda\tau}-1-\tau q(x,t)} \le \frac{\tau}{e^{\lambda\tau}-1-\tau\lambda_0}\max_{\omega_{t_{k+1}}}(f(x,t)e^{-\lambda}\ \ ),$$

and

$$z_i^{k+1} \ge z(x_*, t^{k+1}) \ge \frac{\tau f(x_*, t^k)e^{-\lambda t^k}}{e^{\lambda\tau}-1-\tau q(x_*, t_k)} \ge \min_{\omega_{t_{k+1}}}\frac{\tau f(x,t)e^{-\lambda t}}{e^{\lambda\tau}-1-\tau q(x,t)} \ge \frac{\tau}{e^{\lambda\tau}-1-\tau\lambda_0}\min_{\omega_{t_{k+1}}}(f(x,t)e^{-\lambda t}),$$

which imply that

$$\sup_{\lambda>\bar{\lambda}_0}\left\{\frac{\tau e^{\lambda t_{k+1}}}{e^{\lambda\tau}-1-\tau\lambda_0}\min_{\omega_{t_{k+1}}}(f(x,t)e^{-\lambda t})\right\} \le y_i^{k+1} \le \inf_{\lambda>\bar{\lambda}_0}\left\{\frac{\tau e^{\lambda t_{k+1}}}{e^{\lambda\tau}-1-\tau\lambda_0}\max_{\omega_{t_{k+1}}}(f(x,t)e^{-\lambda t})\right\},$$

i. e., $y_i^{k+1} \in \bar{D}_u$. Thus, the assertion $y_i^{n+1} \in \bar{D}_u$ is proved for all $n = \overline{0, N_0 - 1}$. Moreover, because positivity conditions for the coefficients (7), (8) are fulfilled, the difference scheme (10), (11) is monotone for all values of h and for all $\tau > 0$ if $q_1 \ge 0$, or for $0 < \tau \le -1 / q_1$ if $q_1 < 0$.

Consequently, we have proved the following theorem.

T h e o r e m 3. *The FDS (10), (11) is unconditionally monotone, without any restrictions on the step sizes* τ *and h, and its solution* $y(x, t)$ *remains within the value range* $\bar{D}_u$ *of the exact solution. Furthermore, the following two-sided estimates hold at any grid point* $(x_i, t_n) \in \bar{\omega}$:

$$m_3 \le m_3^n \le y(x_i, t_n) \le m_4^n \le m_4, \tag{13}$$

where m_3^n and m_4^n are constants defined by

$$m_3^n = \sup_{\lambda>\bar{\lambda}_0} e^{\lambda t_n} \min\left\{0, \min_{[0,l]} u_0(x), \min_{[0,t_{n+1}]}(\mu_1(t)e^{-\lambda t}), \min_{[0,t_{n+1}]}(\mu_2(t)e^{-\lambda t}), \frac{\tau}{e^{\lambda\tau}-1-\tau\lambda_0}\min_{\omega_{t_n}}(f(x,t)e^{-\lambda t})\right\},$$

$$m_4^n = \inf_{\lambda > \bar{\lambda}_0} e^{\lambda t_n} \max\left\{0, \max_{[0,l]} u_0(x), \max_{[0,t_{n+1}]} (\mu_1(t)e^{-\lambda t}), \max_{[0,t_{n+1}]} (\mu_2(t)e^{-\lambda t}), \frac{\tau}{e^{\lambda\tau} - 1 - \tau\lambda_0} \max_{\omega_{t_n}} (f(x,t)e^{-\lambda t})\right\},$$

with

$$\bar{\lambda}_0 = \frac{\ln(1 + \tau\lambda_0)}{\tau} \le \lambda_0.$$

R e m a r k. *The estimates obtained in (13) are fully consistent with the estimates (4) for the exact solution of the differential problem (1), (2), and in this sense, one can say that the FDSs inherit the properties of the differential problem.*

Convergence in the grid L_2-norm. In this section, we additionally assume that the exact solution of problem (1), (2) is sufficiently smooth, namely, $u(x,t) \in C^{4,2}(Q_T)$. We use the energy inequality method to obtain estimates of the error and convergence results of scheme (10), (11) in the discrete L_2-norm. Defining the approximation error $\hat{\psi}$ in the interior nodes as

$$\hat{\psi} = -u_t + \varkappa(u)(a(u)\hat{u}_{\bar{x}})_x + b^+(u)a^{(+1)}(u)\hat{u}_x + b^-(u)a(u)\hat{u}_{\bar{x}} + du + \varphi,$$

the grid-function error $z = y - u$ is the solution of the discrete problem

$$z_t = \varkappa(y)(a(y)\hat{z}_{\bar{x}})_x + \varkappa(y)((a(y) - a(u))\hat{u}_{\bar{x}})_x + (\varkappa(y) - \varkappa(u))(a(u)\hat{u}_{\bar{x}})_x + b^+(y)a^{(+1)}(y)\hat{z}_x +$$
$$+ b^-(y)a(y)\hat{z}_{\bar{x}} + (b^+(y)a^{(+1)}(y) - b^+(u)a^{(+1)}(u))\hat{u}_x + (b^-(y)a(y) - b^-(u)a(u))\hat{u}_{\bar{x}} + dz + \hat{\psi},$$

with initial and boundary conditions

$$z(x,0) = 0, \quad x \in \bar{\omega}_h, \quad z(0,t) = z(l,t) = 0, \quad t \in \omega_\tau.$$

It is easy to see that $\hat{\psi} = O(h^2 + \tau)$ in all the nodes. We now define the following inner products and the corresponding norms

$$(u,v) = \sum_{x \in \omega_h} hu(x)v(x), \quad \|u\| = \sqrt{(u,u)},$$

$$(u,v] = \sum_{x \in \omega_h^+} hu(x)v(x), \quad \|u]| = \sqrt{(u,u]},$$

where $\omega_h^+ = \omega_h \cup \{l\}$.

The following assertion holds.

T h e o r e m 4. *The solution of the FDS (10), (11) converges to the exact solution of differential problem (1), (2), and the following estimate of the method accuracy holds*

$$\|\hat{z}\| \le C(h^2 + \tau),$$

C being a positive constant independent of the discretization parameters.

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