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TUNNELING THROUGH A SMOOTH PARABOLIC BARRIER OF FINITE HEIGHT

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The smooth barrier of finite height and variable shape is constructed by means of joining the central inverted parabolic potential and two side parabolic potentials. The problem of tunneling through this barrier is solved exactly. The dependence of the transmission coefficient on energy is presented. The real and imaginary components of wave functions are shown.

Keywords: tunneling, parabolic barrier, transmission coefficient.

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ТУННЕЛИРОВАНИЕ ЧЕРЕЗ ГЛАДКИЙ ПАРАБОЛИЧЕСКИЙ БАРЬЕР КОНЕЧНОЙ ВЫСОТЫ

(Представлено членом-корреспондентом Л. М. Томильчиком)

Гладкий барьер конечной высоты и варьируемой формы построен с помощью соединения центрального перевернутого параболического потенциала и двух боковых параболических потенциалов. Задача о туннелировании через этот барьер решена точно. Представлена зависимость коэффициента прохождения от энергии. Показаны реальные и мнимые составляющие волновых функций.

Ключевые слова: туннелирование, параболический барьер, коэффициент прохождения.

Introduction. Tunneling of a particle through a potential barrier is one of the important phenomena of quantum mechanics. The interest in this problem ranges from various branches of physics to chemistry. However, a limited number of potentials can be solved exactly [1]. Among them there are several parabolic potentials.

First of all, it is an inverted harmonic oscillator [2; 3] $V(q) = -kq^2$ which is too much idealized in our opinion. A truncated inverted parabolic potential [4; 5]

$$V(q) = V_0 \begin{cases} 1 - \dfrac{q^2}{q_0^2}, & |q| < q_0, \\ 0, & |q| > q_0 \end{cases} \tag{1}$$

is more realistic for simulation of physical process. The potential function (1) has a finite height V_0 and its first derivative is discontinuous at the points $q = \pm q_0$.

At last, it should be noted that a double oscillator model [6; 7] $V(q) = V_0(|q| - q_0)^2 / q_0^2$ of double-well potential can be modified to a single parabolic barrier

$$V(q) = V_0 \begin{cases} \dfrac{(|q| - q_0)^2}{q_0^2}, & |q| < q_0, \\ 0, & |q| > q_0. \end{cases} \tag{2}$$

This quadratic potential of a finite height consists of two parabolas which meet with discontinuous slope at the point $q = 0$.

Both potentials (1) and (2) are not smooth. At the same time it is possible to construct a smooth potential with the help of a inverted parabola in the central region and two shifted parabolas in both side regions. The new potential function is of the form

$$V(q) = V_0 \begin{cases} 1 - \dfrac{q^2}{gq_0^2}, & |q| < gq_0, \\ \dfrac{(|q| - q_0)^2}{(1-g)q_0^2}, & gq_0 < |q| < q_0, \\ 0, & |q| > q_0. \end{cases} \tag{3}$$

Here $0 < g < 1$. The second derivative of the function (3) is discontinuous at the points $q = \pm q_0$ and $q = \pm gq_0$. However, both the function (3) and its first derivative are continuous. The considered potential barrier coincides with (1) if $g = 1$ and with (2) if $g = 0$. The presence of a varied parameter g allows to change a shape of barrier (3) in the wide range. Due to this circumstance the proposed potential becomes very helpful for simulation of tunneling phenomena.

Analytical solution. We are interesting in solving the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + V(q) \right) \Psi(q) = E\Psi(q),$$

where $V(q)$ takes the form (3). It is convenient to introduce dimensionless quantities

$$x = \sqrt{\frac{2mV_0}{\hbar^2}}\, q, \quad x_0 = \sqrt{\frac{2mV_0}{\hbar^2}}\, q_0, \quad e = \frac{E}{V_0}.$$

The transformed Schrödinger equation is given as

$$\left(-\frac{d^2}{dx^2} + v(x) \right) \psi(x) = e\psi(x) \tag{4}$$

with the scaled potential

$$v(x) = \begin{cases} 1 - \dfrac{x^2}{gx_0^2}, & |x| < gx_0, \\ \dfrac{(|x| - x_0)^2}{(1-g)x_0^2}, & gx_0 < |x| < x_0, \\ 0, & |x| > x_0. \end{cases} \tag{5}$$

The shape of $v(x)$ is shown in fig. 1 for different values of g when $x_0 = 2$. Here and in subsequent figures we use dotted lines for $g = 0.1$, solid lines for $g = 0.5$ and dashed lines for $g = 0.9$.

The simplicity of the considered potential (5) permits to find the exact solutions of Eq. (4) in five regions. The wave function is represented in the following way

$$\psi(x) = \begin{cases} \exp(i\sqrt{e}\,x) + A_1 \exp(-i\sqrt{e}\,x), & x < -x_0, \\ A_2\, y_{s1}(z_s) + A_3\, y_{s2}(z_s), & -x_0 < x < -gx_0, \\ A_4\, y_{c1}(z_c) + A_5\, y_{c2}(z_c), & -gx_0 < x < gx_0, \\ A_6\, y_{s1}(z_s) + A_7\, y_{s2}(z_s), & gx_0 < x < x_0, \\ A_8 \exp(i\sqrt{e}x), & x > x_0. \end{cases}$$

There are the incident and reflected waves in the region $x < -x_0$ and there is the transmitted wave in the region $x > x_0$. It is not hard to show that the particular solutions in the region $-x_0 < x < x_0$ are expressed in terms of the confluent hypergeometric functions [8]. In the side regions $-x_0 < x < -gx_0$ and $gx_0 < x < x_0$, the explicit solutions are given by formulas

$$y_{s1}(z_s) = e^{-z_s^2/4} M\left(\frac{a_s}{2} + \frac{1}{4}, \frac{1}{2}, \frac{z_s^2}{2}\right),$$

$$y_{s2}(z_s) = z_s e^{-z_s^2/4} M\left(\frac{a_s}{2} + \frac{3}{4}, \frac{3}{2}, \frac{z_s^2}{2}\right),$$

$$z_s(x) = \left(\frac{2}{x_0}\right)^{1/2} \frac{(|x| - x_0)}{(1-g)^{1/4}}, \quad a_s = -\frac{\sqrt{1-g}}{2} x_0 e.$$

In the central region $-gx_0 < x < gx_0$, we have the following solutions

$$y_{c1}(z_c) = \frac{1}{2}\left\{ e^{-iz_c^2/4} M\left(-\frac{ia_c}{2} + \frac{1}{4}, \frac{1}{2}, \frac{iz_c^2}{2}\right) + e^{iz_c^2/4} M\left(\frac{ia_c}{2} + \frac{1}{4}, \frac{1}{2}, -\frac{iz_c^2}{2}\right)\right\},$$

$$y_{c2}(z_c) = \frac{z_c}{2}\left\{ e^{-iz_c^2/4} M\left(-\frac{ia_c}{2} + \frac{3}{4}, \frac{3}{2}, \frac{iz_c^2}{2}\right) + e^{iz_c^2/4} M\left(\frac{ia_c}{2} + \frac{3}{4}, \frac{3}{2}, -\frac{iz_c^2}{2}\right)\right\},$$

$$z_c(x) = \left(\frac{2}{x_0}\right)^{1/2} \frac{x}{g^{1/4}}, \quad a_c = \frac{\sqrt{g}}{2} x_0 (1-e).$$

It should be stressed that these solutions are real.

By joining the wave function and its first derivative smoothly at four points $x = -x_0, -gx_0, gx_0, x_0$ we obtain the system of eight algebraic equations for eight coefficients A_i. It is easily to solve this system but the solutions are very cumbersome. Therefore we represent only one coefficient

$$A_8 = \frac{-\left(\frac{4}{x_0^2 g}\right)^{1/4} \exp\left(-2i\sqrt{e}\, x_0\right)}{\left(\frac{x_0^2(1-g)}{4}\right)^{1/4} (f_{11} f_{22} + f_{12} f_{21}) + i\sqrt{e}\left(\frac{x_0\sqrt{1-g}}{2} f_{12} f_{22} - \frac{1}{e} f_{11} f_{21}\right)},$$

where we use notations

$$f_{ij} = \sqrt{\frac{2}{x_0}}\left((1-g)^{-1/4}\, \bar{y}_{ci} \bar{y}'_{sj} - g^{-1/4}\, \bar{y}_{sj} \bar{y}'_{ci}\right), \quad i = 1, 2, \quad j = 1, 2,$$

$$\bar{y}_{ci} = y_{ci}(\bar{z}_c), \quad \bar{y}'_{ci} = \frac{dy_{ci}(\bar{z}_c)}{d\bar{z}_c}, \quad \bar{z}_c = z_c(gx_0) = \sqrt{2x_0}\, g^{3/4},$$

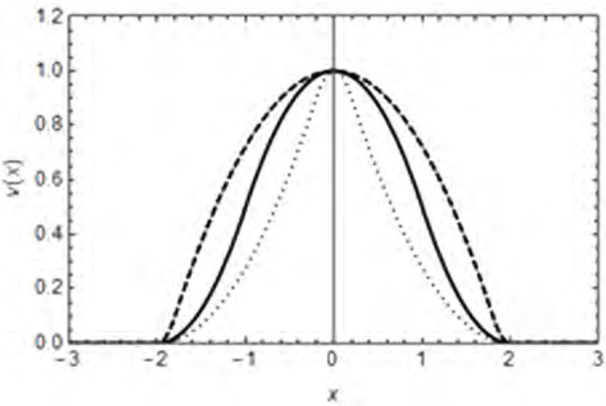

Fig. 1. The scaled potential $v(x)$ for different values of g

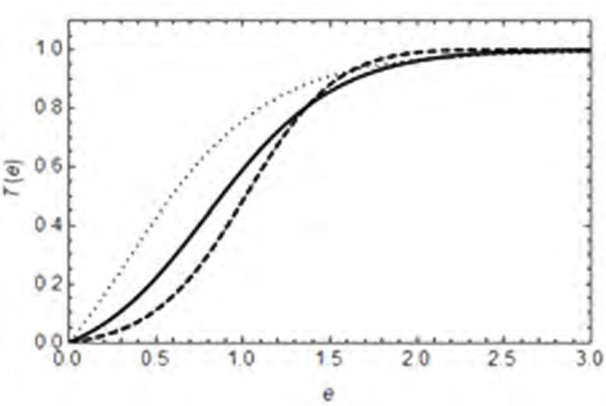

Fig. 2. Dependence of T on e for $x_0 = 2$

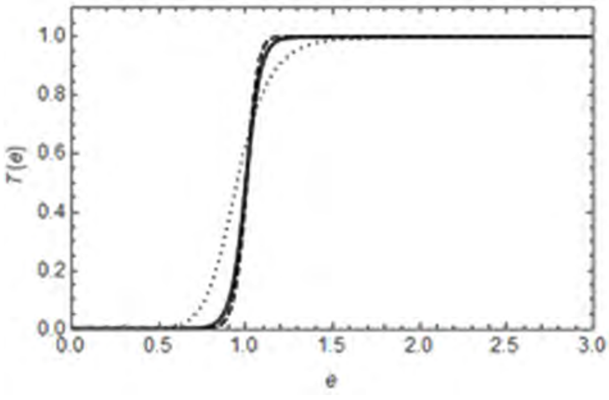

Fig. 3. Dependence of T on e for $x_0 = 10$

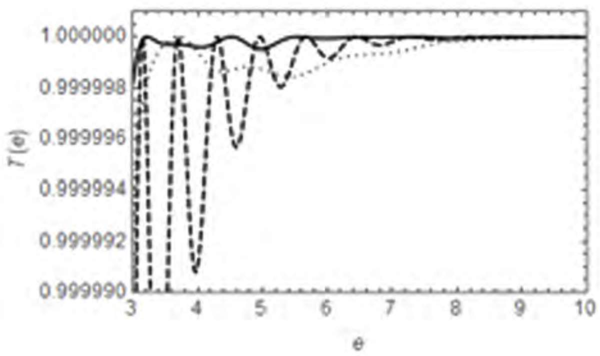

Fig. 4. Dependence of T on large e for $x_0 = 10$

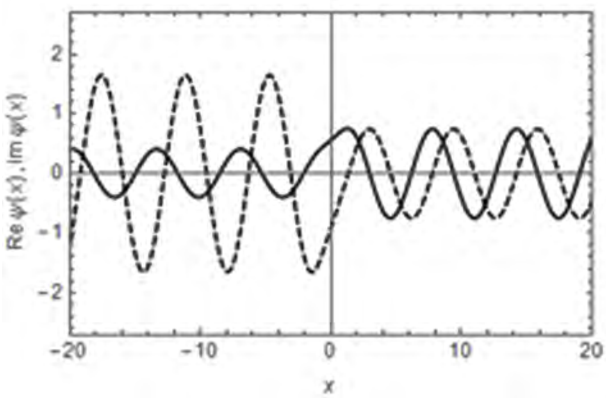

Fig. 5. Wave function for $x_0 = 2$

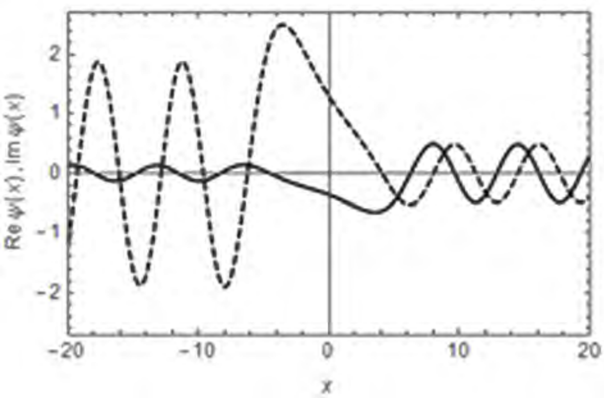

Fig. 6. Wave function for $x_0 = 10$

$$\bar{y}_{sj} = y_{sj}(\bar{z}_s), \quad \bar{y}'_{sj} = \frac{dy_{sj}(\bar{z}_s)}{d\bar{z}_s}, \quad \bar{z}_s = z_s(gx_0) = -\sqrt{2x_0}\,(1-g)^{3/4}.$$

The square of the absolute value of A_8 is the transmission coefficient T for the proposed barrier (3). The final exact expression is

$$T = |A_8|^2 = \frac{1}{1 + \frac{e x_0 \sqrt{g}}{2}\left(\frac{x_0\sqrt{1-g}}{2} f_{12} f_{22} + \frac{1}{e} f_{11} f_{21}\right)^2}.$$

Graphic illustrations. The dependence of the transmission coefficient T on a scaled energy e is given in fig. 2 for $x_0 = 2$ and in fig. 3 for $x_0 = 10$ at different values of g. It should be noted that T can be equal to 1 at selected values of e for $e > 1$ (or $E > V_0$). This property is demonstrated in fig. 4 for $x_0 = 10$. For example, $T = 1$ at $e = 1.83062, 2.48069, 3.22345, 4.51937$ for $1 < e < 5$ if $g = 0.5$ and $x_0 = 10$.

At last, the real (solid lines) and the imaginary (dashed lines) components of wave functions are represented in fig. 5 for $x_0 = 2$ and in fig. 6 for $x_0 = 10$ at $e = 0.95$ and $g = 0.5$.

Conclusion. The proposed parabolic potential extends a short list of exactly solvable models that describe tunneling through barriers. The variable shape of considered barrier gives the wide possibilities to simulate the tunneling phenomena. In the present paper, we examined a symmetric potential but it is not hard to construct an asymmetric smooth parabolic potential. In addition to the case of a single barrier it will be desirable to investigate the system of the several barriers.

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