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CAPTURING A VISCOPLASTIC LIQUID BY A MOVING VERTICAL PLATE

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The liquid capture by a moving surface is the most widespread process in chemical engineering along with calendaring, extrusion moulding, pouring, and pressure moulding. The theoretical analysis of the medium capture by a moving surface, which allows revealing the fundamental physical principles and mechanisms of the process over the entire withdrawal speed range realized in practice, was performed for Newtonian, non-Newtonian, and viscoplastic liquids. However, such an analysis of the withdrawal of viscoplastic liquids with a finite yield was not made because of the features of these liquids. Shear flow of viscoplastic liquid is possible only after the stress exceeds its yield. This fact causes serious mathematical difficulties in stating and solving the problem. In the proposed work, such a theory is being developed for viscoplastic liquids.

Keywords: viscoplastic liquid, liquid withdrawal velocity, liquid layer width, static and dynamic menisci.

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УВЛЕЧЕНИЕ ВЯЗКОПЛАСТИЧЕСКОЙ ЖИДКОСТИ ДВИЖУЩЕЙСЯ ВЕРТИКАЛЬНО ПЛАСТИНОЙ

(Представлено академиком О. Г. Пенязьковым)

Захват жидкости движущейся поверхностью является наиболее распространённым процессом в химической технологии наряду с каландрованием, экструзионным формованием, заливкой, формованием под давлением. Теоретический анализ увлечения среды движущейся поверхностью, позволяющей вскрыть основные физические принципы и механизмы процесса во всем диапазоне скоростей извлечения, реализуемом на практике, был проведен для ньютоновских, нелинейновязких, вязкопластичных жидкостей. Однако такой анализ по увлечению вязкопластичных жидкостей, обладающих конечным пределом текучести, проведен не был в силу специфических особенностей этих жидкостей. Для вязкопластичной жидкости сдвиговое течение возможно лишь после того как напряжение превысит предел текучести. Данное обстоятельство вносит серьезные математические трудности при постановке и решении задачи. В предлагаемой работе такая теория развивается для вязкопластичных жидкостей.

Ключевые слова: вязкопластические жидкости, скорость извлечения жидкости, ширина слоя жидкости, статические и динамические мениски.

Introduction. As one of the continuous established technologies of moulding and processing materials the process of capturing liquid by a moving surface is the most widespread process in chemical engineering along with calendaring, extrusion moulding, pouring, and pressure moulding. Determination of the thickness of the film deposited on the surface of paper, polymer, metal, and fabric withdrawn from a solution is of substantial significance for such technological processes as:

a) application of different coatings – protective, decorative, special (light- and magnet-sensitive, electrically insulated, clay, release, abrasion, anti-friction, etc.);

b) drying in contact roller machines (rotating heated drum capture of a medium film with its subsequent drying);

c) crystallization in knife-discharge drum crystallizers (a solidified layer thickness is determined through the capture of a melt film by the drum surface);

d) dispersion by submerged mechanical sprayers (the rotating discs partially submerged into liquid withdrawn in the form of a film, followed by its dispersion);

e) filtration in drum and disc vacuum-filters, etc.

The works [1–3] were concerned with the theoretical analysis of capture of a medium by a moving surface. It allowed one to uncover the basic physical principles and the mechanisms of the capture process over the entire range of the withdrawal speed realized in practice and at a different rheological state of liquid (Newtonian, non-Newtonian, viscoelastic).

However, such an analysis of the capture of viscoplatic liquids with a finite yield was not made because of the features of these liquids that are different from those of Newtonian and non-plastic non-Newtonian media. Shear flow of the viscoplastic liquid is possible only after the film stress exceeds its yield. This fact causes serious mathematical difficulties in stating and solving the problem.

Problem statement. To describe the rheological behavior of a viscoplastic medium, Shvedov–Bingham's linear model for viscous shear stress has found widest use [4]

$$\begin{aligned} &\tau = \tau_0 \operatorname{sign}\frac{du}{dy} + \mu_P \frac{du}{dy}, && |\tau| > \tau_0, \\ &\frac{du}{dy} = 0, && |\tau| \le \tau_0. \end{aligned} \qquad (1)$$

Here u is the liquid velocity, τ_0 is the ultimate shear stress (yield), and μ_P is the plastic viscosity. Writing (1) stands for the demand of the same signs of τ and du / dy that follows from the essence of the phenomenon considered.

Consider the liquid being into the bath, from which an infinite plate is withdrawn vertically upwards with a constant speed U (Fig. 1). The thickness of the layer captured by the plate wall decreases with increasing distance from the horizontal liquid surface and asymptotically tends to a constant value of h_0. Because of the gravity, this plate captures only some amount of the liquid put in motion by it. Therefore, the stagnation line h_S is seen in the direction to the free surface where the layer velocity is equal to zero. As a result, the speed of moving the free surface of the film captured by the plate increases from zero in the stagnation line to its maximum value in the region of the constant film thickness h_0. The stagnation line separates the near-wall zone of the liquid captured by the plate from the bath zone. For these zones the equations responsible for the shape of the free surface can be obtained and the solutions can then be joined. This will allow the thickness of the entrained film to be determined.

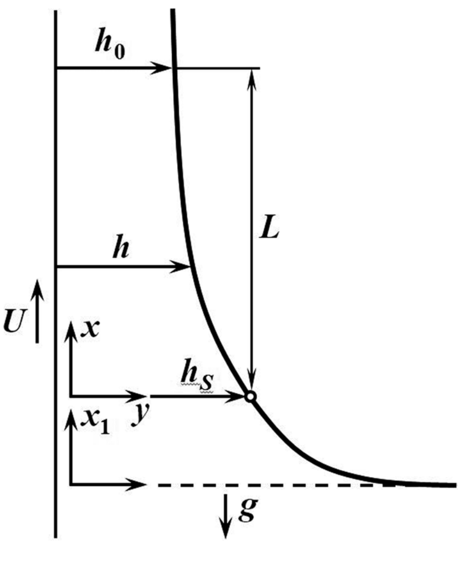

Fig. 1. Viscoplastic liquid flow scheme

Let the stagnation line be the coordinate origin, the x axis be in the direction of movement of the plate and the y axis be perpendicular to it. Define the region of dynamic meniscus of liquid as the flow area bounded from below by the plane perpendicular to the wall and passing through the stagnation line and from above moving into the region of the constant thickness h_0. In this region the film thickness h is determined by the ratio of internal friction forces, surface tension, gravity, and inertia. From the physical considerations it is clear that the characteristic dynamic meniscus length L much exceeds the film thickness h_0. Then the small parameter $\varepsilon = h_0 / L << 1$ arises quite naturally. This means that the change in the flow characteristics along the x axis is weaker than in the transverse y direction, i. e., the derivatives with respect to the y axis will be much larger than those with respect to the x axis.

As a result of the dynamic meniscus of the non-Newtonian liquid, correct to the terms of the order of ε^2, we obtain

$$\frac{\partial \tau}{\partial y} - \rho g + \sigma \dddot{h} = 0, \quad p - p_0 = -\sigma \ddot{h} \quad (y = h),$$
$$u = U\ (y = 0), \quad \tau = 0\ (y = h). \tag{2}$$

Hereinafter, the following notations are taken: $\dot{h} = dh / dx$, $\ddot{h} = d^2h / dx^2$, $\dddot{h} = d^3h / dx^3$. Due to a high consistency of non-Newtonian media, the inertia terms can be ignored. Present the equation of continuity in integral form

$$Q = \int_0^h u dy = \text{const.} \tag{3}$$

Integrating equation of motion (2) with the boundary condition satisfied yields

$$\tau(y) = -(\rho g - \sigma \dddot{h})(h - y).$$

Choose the rheological equation for the viscoplastic liquid in the form of Shvedov–Bingham's classical model (1). Considering that $du / dy < 0$ for the shear flow region we have

$$-\tau_0 + \mu_P \frac{\partial u}{\partial y} = -(\rho g - \sigma \dddot{h})(h - y). \tag{4}$$

As equation (4) shows, across the film the shear stress changes continuously and linearly, starting with a maximum value at the plate wall to a zero value at the free surface. At some value of the ordinate $y = \delta$, the magnitude $\tau = \tau_0$ can be attained. Bearing in mind the second condition of rheological equation of state (1) and equation (4) we obtain

$$\tau_0 = (\rho g - \sigma \dddot{h})(h - \delta).$$

Hence, for the viscoplastic liquid flow zone $\tau > \tau_0$ $(0 \le y < \delta)$

$$\delta = h - \tau_0 / (\rho g - \sigma \dddot{h}) \tag{5}$$

and for the liquid flow quasi-solid zone $\tau \le \tau_0$ $(\delta \le y \le h)$

$$\Delta = h - \delta = \tau_0 / (\rho g - \sigma \dddot{h}). \tag{6}$$

Integrating equation (4) with respect to y yields the velocity distribution across the film in the viscoplastic liquid flow zone

$$u = U + \frac{\tau_0}{\mu_P} y - \frac{1}{\mu_P} (\rho g - \sigma \dddot{h}) \left(hy - \frac{y^2}{2} \right). \tag{7}$$

In turn, for the velocity of the liquid flow quasi-solid zone the substitution of equation (7) at $y = \delta$ into expression (5) arrives at:

$$u_0 = U + \frac{\tau_0}{\mu_P} h - \frac{h^2}{2\mu_P} (\rho g - \sigma \dddot{h}) - \frac{\tau_0}{2\mu_P} / (\rho g - \sigma \dddot{h}). \tag{8}$$

Define the liquid flow velocity in the film using formula (3):

$$Q = Uh + \frac{\tau_0 h^2}{2\mu_P} - \frac{h^3}{3\mu_P} (\rho g - \sigma \dddot{h}) - \frac{\tau_0^3}{6\mu_P} / (\rho g - \sigma \dddot{h})^2. \tag{9}$$

The last equation is valid over the range of the film thickness $h(x)$ from h_0 to h_S.

Find the position of the stagnation line h_S, assuming that $u_0 = 0$ and $h = h_S$ in expressions (8) and (9). Then

$$h_S = 3 \frac{Q}{U} \left(1 + \frac{3}{2} \frac{Q}{U} \frac{\tau_0}{\mu_P U} \right). \tag{10}$$

Equation (9) assigns the film thickness $h(x)$ through the predetermined quantities U, ρ, g, σ, μ_P, τ_0. Since the general solution to this nonlinear differential equation cannot be obtained, consider some particular cases. To do this, equation (9) with regard to formulas (5) and (6) is written in equivalent form:

$$Q = Uh - \frac{\tau_0 \delta^2}{2\mu_P}\left(\frac{1-\delta/3h}{1-\delta/h}\right); \qquad Q = Uh - \frac{\tau_0 h^3}{3\mu_P \Delta}\left(1 - \frac{3\Delta}{2h} + \frac{\Delta^3}{2h^3}\right). \tag{11}$$

The thickness of the quasi-solid movement zone is much larger than that of the viscoplastic movement zone. Let the condition $\delta / h << 1$ be satisfied in the first equation of (11), i. e., the quasi-solid movement zone thickness Δ is much larger than the shear viscoplastic liquid zone thickness δ. Then the main role in this equation will be played by the plastic factor τ_0. As limiting cases, the rheological equation of state of viscoplastic liquid (1) indeed contains the equations of state of perfectly viscous liquid $\tau_0 = 0$ and perfectly plastic substance $\mu_P \frac{du}{dy} = 0$. It is natural that depending on the relation between τ_0 and $\mu_P \frac{du}{dy} = 0$, the properties of viscoplastic liquid will approach those either of viscous liquid or of perfectly plastic substance.

Below there is the condition, under which the requirement $\delta / h << 1$ is satisfied, and while in the first equation of (11) the term δ / h will be neglected in comparison to unity. Then

$$\delta = \sqrt{\frac{2\mu_P}{\tau_0} Uh \left(1 - \frac{Q}{Uh}\right)}.$$

Substituting this formula into equation (5) gives

$$\frac{\sigma}{\rho g}\dddot{h} = 1 - \frac{\tau_0}{\rho g h}\left[1 - \sqrt{\frac{2\mu_P U}{\tau_0 h}\left(1 - \frac{Q}{Uh}\right)}\right]^{-1}. \tag{12}$$

For the region of the constant film thickness, when $h = h_0$, all derivatives with respect to the x coordinate are equal to zero. As a result,

$$\frac{Q}{Uh} = 1 - \frac{\tau_0 h_0}{2\mu_P U}\left(1 - \frac{\tau_0}{\rho g h_0}\right)^2, \tag{13}$$

and upon reduction to dimensionless variables

$$\xi = \frac{x}{h_0},\ H = \frac{h}{h_0},\ Ca = \frac{U\mu_P}{\sigma},\ D = h_0\sqrt{\frac{\rho g}{\sigma}},\ B = \frac{\tau_0}{\sqrt{\rho g \sigma}} \tag{14}$$

equation (12) assumes the form

$$\frac{d^3 H}{d\xi^3} = D^2 - BD\left[H - \sqrt{\frac{2Ca}{BD}(H-1) + (1 - B/D)^2}\right]^{-1}. \tag{15}$$

The ultimate value of the film thickness h_0 is attained asymptotically at a sufficiently large value of the x coordinate. To sufficient accuracy, it can be assumed that $H \to 1, \frac{dH}{d\xi} \to 0, \frac{d^2 H}{d\xi^2} \to 0$ as $\xi \to \infty$. Equation (15) contains one more unknown magnitude D (or h_0 in dimensionless form) related to the liquid flow velocity by expression (13). To calculate it, it is necessary to find the shape of the liquid surface below the stagnation line and then to fit it to the shape of the dynamic meniscus in the stagnation line. The condition to fit the shapes will be that missing condition that will allow h_0 to be calculated.

Now consider the liquid surface extending to the right of the stagnation line. By assumption, this surface obeys the equations of capillary statics, in particular the Laplace equation:

$$p - p_0 = -\sigma \frac{d^2 h}{dx_1^2}\left[1 + \left(\frac{dh}{dx_1}\right)^2\right]^{-3/2}. \tag{16}$$

Here the x_1 axis coincides with the x axis, but taken from the horizontal surface of the liquid poured into the bath (Fig. 1). Call the zone below the stagnation line the region of the static meniscus of the liquid.

It is obvious that in the stagnation line, the liquid pressure determined on the side of both the dynamic and static regions must be the same. Then, following from equations (2) and (16) the condition to join the solutions for the dynamic and static regions can be found:

$$\ddot{h} = \frac{d^2h}{dx^2} = \frac{d^2h}{dx_1^2}\left[1+\left(\frac{dh}{dx_1}\right)^2\right]^{-3/2} \quad \text{at } h = h_S. \quad (17)$$

The position of the stagnation line h_S is calculated by substituting expression (13) into formula (10).

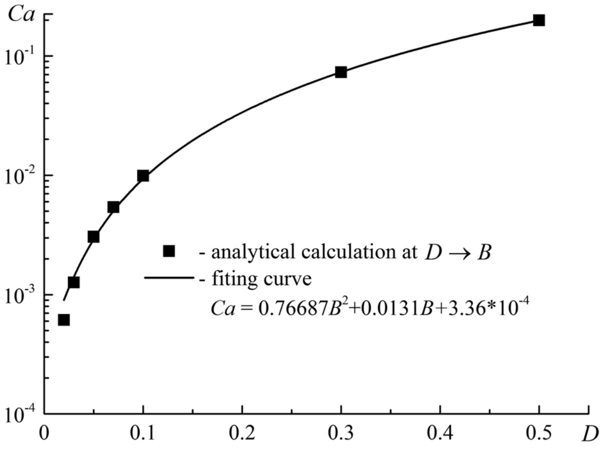

Fig. 2. Withdrawal velocity *Ca* vs. film width *D* at $D \to B$

The calculation results (Fig. 2) demonstrate that there exists some dimensionless withdrawal speed, Ca^*, of the plate, at which the dimensionless film thickness D becomes equal to the plastic factor B and the entrained film thickness $h_0 = \dfrac{\tau_0}{\rho g}$ does not depend on surface tension.

The case of no viscoplastic movement zone. Consider the case when the dimensionless withdrawal speed Ca is smaller than Ca^*. Then the viscoplastic flow region is absent, i. e., $\delta = 0$, and the equation for the dynamic meniscus of the liquid will be of the form

$$\frac{\sigma}{\rho g}\dddot{h} = 1 - \frac{\tau_0}{\rho g h}. \quad (18)$$

In this case, the dynamic meniscus (Fig. 3) can be divided into two zones – upper A with no plastic deformation and – lower C with plastic deformation. When liquid deformation is absent in zone A, the liquid layer moves in it as a single whole with the plate moving upwards with the latter. In view of this, from the condition of the constant liquid flow rate $Q = \text{const}$ and the constant liquid velocity $u = \text{const} = U$ we have that the thickness of the liquid layer at any point of zone A is the same and equal to $h_0 = Q/U = \text{const}$. Thus, in zone A the film thickness is constant, whereas in the previous considered case it has tended asymptotically to a constant value.

It is obvious that at boundary F of zones A and C, the conditions $h_A = h_C = h_0$, $\dot{h}_A = \dot{h}_C = 0$, and $\ddot{h}_A = \ddot{h}_C = 0$ are valid. The last condition follows from the requirement of pressure continuity in the layer.

As for the definition of the liquid surface shape in zone C and the condition to join this shape and the static meniscus shape, the following should be emphasized. The film thickness h in equation (18) obviously changes from a minimum value of h_0 at point F of zone C to a maximum value equal to $h_{\max} = h_S = \dfrac{\tau_0}{\rho g}$. Here $h_0 < \dfrac{\tau_0}{\rho g}$, since in the opposite case the value of the derivative of $\dddot{h}$ at point F will be positive. A maximum value of the film thickness $h_{\max}$ separates the zone of the dynamic meniscus from the static one, i. e., in fact, it is the stagnation line. The above method of joining the two menisci then remains valid.

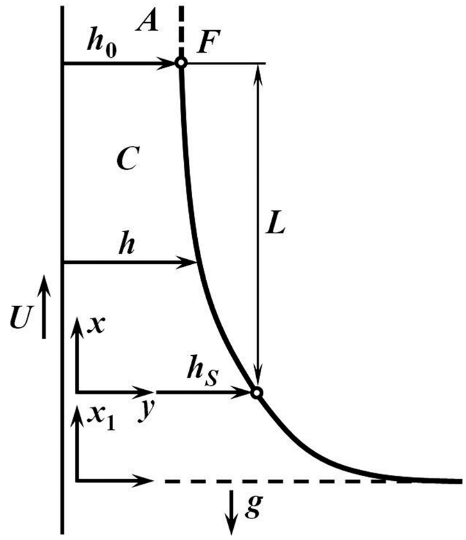

Fig. 3. Flow scheme for the case of no viscoplastic flow zone

In dimensionless variables (14), the equation of the dynamic meniscus and the position of the stagnation line are of the form

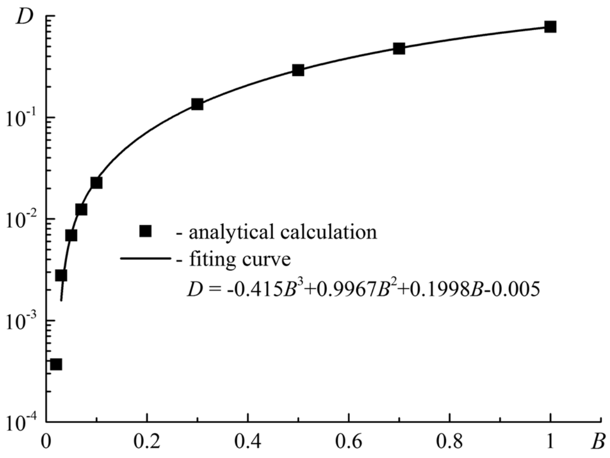

Fig. 4. Film width D vs. plastic factor B for the case of no viscoplastic flow zone

$$\frac{d^3H}{d\xi^3} = D^2 - \frac{BD}{H}, \;\; H_S = B/D.$$

Hence it follows that the entrained liquid film thickness h_0 does not depend on the withdrawal speed, Ca, of the plate. The calculation results in Fig. 4 confirm this fact.

The thickness of the quasi-solid flow zone is commensurable with the viscoplastic flow zone thickness. Assume that in the second equation of (11) the condition $1 - \frac{3\Delta}{2h} >> \frac{\Delta^3}{2h^3}$ is satisfied, then neglecting the term $\frac{\Delta^3}{2h^3}$, for the liquid flow rate in the film we have:

$$Q = Uh - \frac{\tau_0 h^3}{3\mu_P \Delta}\left(1 - \frac{3\Delta}{2h}\right). \qquad (19)$$

To find the condition for feasibility of equation (19), it is assumed that $1 - \frac{3\Delta}{2h} \geq 10\frac{\Delta^3}{2h^3}$. This means that $\frac{\Delta}{h} < \frac{1}{2}$. Thus, equation (19) is valid for quite a common case when the thickness of the quasi-solid flow zone is of the same order as the thickness of the viscoplastic flow zone. Having used formula (6), reduce equation (19) to the form

$$Q = Uh + \frac{\tau_0}{2\mu_P}h^2 - \frac{\rho g}{3\mu_P}h^3 + \frac{\sigma}{3\mu_P}h^3\dddot{h}. \qquad (20)$$

The withdrawn layer thickness h enough tends asymptotically to a constant value of h_0 at a sufficiently large distance from the liquid surface in the bath (as $x \to \infty$) (Fig. 1). At a time, all derivatives of h with respect x tend to zero. Substitution of the appropriate quantities into equation (20) yields

$$Q = Uh_0 + \frac{\tau_0}{2\mu_P}h_0^2 - \frac{\rho g}{3\mu_P}h_0^3. \qquad (21)$$

Combining equalities (20) and (21), and also using the dimensionless variables (14), for the dynamic meniscus we have

$$\frac{H^3}{3Ca}\frac{d^3H}{d\xi^3} = (1-H)\left[1 + \frac{BD}{2Ca}(1+H) - \frac{D^2}{3Ca}(1+H+H^2)\right]. \qquad (22)$$

As previously, to solve the problem stated – to determine the layer thickness h_0 – it is necessary to use condition (17) of joining the solutions for the dynamic and static menisci in the stagnation line h_S. The position of the latter is found by substituting expression (21) into formula (10). The calculation results are shown in Fig. 5. It should be noted that in deriving equation (22) it was assumed that $\frac{\Delta}{h} < \frac{1}{2}$. For sufficiently large

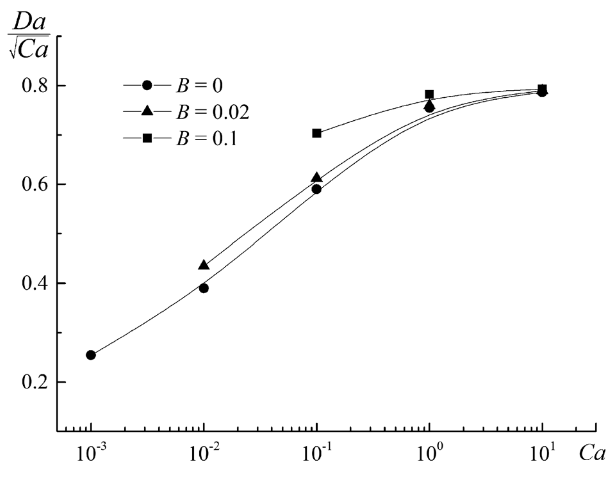

Fig. 5. Reaching the asymptotic value of $D/\sqrt{Ca}$ at increase in the withdrawal speed Ca and at different values of the plastic factor B

distances from the static meniscus this condition is re-written as $\frac{\Delta_0}{h_0} < \frac{1}{2}$ where $\Delta_0 = \frac{\tau_0}{\rho g}$ according to formula (6). The reduction of the obtained condition to the dimensionless form through D and B is indicative of the fact that equation (22) can be used for the case when $2B < D$. From Fig. 5 it is seen that at sufficiently large withdrawal speed, *Ca*, of the plate, the film thickness becomes independent of surface tension and is assigned only by friction and gravity forces.

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